

Zeno's Arrow: A Mathematical Speculation

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"If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless" – Aristotle on Zeno

"Time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles" – Aristotle's objection

"Instants are not parts of time, for time is not made up of instants any more than a magnitude is made of points, as we have already proved. Hence it does not follow that a thing is not in motion in a given time, just because it is not in motion in any instant of that time" - Saint Thomas Aquinas commenting on Aristotle's objection of Zeno's Paradox of the Arrow in Flight

Introduction: Let a set of points A in the complex plane \mathcal{C} be considered an event that will change over time. For each z in A an *event evolution function* will transform the original event into its evolved form after a set period of time. An evolution function (EF), $z(t)$ differentiable and thus continuous in t , will describe an instantaneous rate of change at time t_0 through its derivative $z'(t_0)$. The only time the derivative equals 0 is when the EF is "flat" i.e., there is no instantaneous change. If this were the case throughout a time interval $P = [0,1]$ there would be no change over P . However, a different philosophical perspective might suggest that whereas a change does take place over P , over infinitesimal intervals the change is in fact 0, and still the original event changes over P in a "continuous" fashion. It is the purpose of this note to describe one way this might easily be modeled mathematically by using *Tannery's Series* that do *not* conform to Tannery's Theorem [1].

Proposition: Consider functions of a complex variable $g_{k,n}(z) = z + \phi_{k,n}(z)$ where

$z \in S \Rightarrow g_{k,n}(z) \in S$ and $\lim_{n \rightarrow \infty} \phi_{k,n}(z) = 0$ for all $1 \leq k \leq n$ and all $z \in S$. Thus

$g_{k,n}(z) \rightarrow z$, for each k as $n \rightarrow \infty$. Partition the time interval $P=[0,1]$ into n equal subintervals of

the form $\left[\frac{k-1}{n}, \frac{k}{n} \right]$. Apply $g_{1,n}(z)$ to change an *event*, z , over the interval $\left[0, \frac{1}{n} \right]$, then apply

$g_{2,n}(g_{1,n}(z)) = g_{2,n} \circ g_{1,n}(z)$ over $\left[\frac{1}{n}, \frac{2}{n} \right]$, etc. The total event evolution over P may then be written

$G_{n,n}(z)$, where $G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \dots \circ g_{1,n}(z)$. To continue the process, simply allow $n \rightarrow \infty$.

If $\lim_{n \rightarrow \infty} G_{n,n}(z) = G(z)$ exists, then $G(z)$ is the "*continuous*" evolution of event z (or set A) over P , and at each "instant" the change of (each part of) the event is 0.

Example 1: Set $g_{k,n}(z) = z + \frac{k}{n^2}C$. Then $G_{n,n}(z) \rightarrow G(z) = z + C \int_0^1 w dw = z + \frac{C}{2}$, a translation. In a larger sense, the set A becomes the set $A + C/2$.

Observe the following:

$$G_{n,n}(z) = z + \varphi_{1,n}(z) + \varphi_{2,n}(G_{1,n}(z)) + \varphi_{3,n}(G_{2,n}(z)) + \dots + \varphi_{n,n}(G_{n-1,n}(z)) ,$$

Which is a *Tannery Series*, and which would be of little consequence if the classical *Tannery's Theorem* [1] applied, for then $G(z) \equiv z$ and there would have been no change, reflecting – under the notions of classical calculus – an instant rate of change of 0 at each point of the entire interval P.

Thus we go outside the realm of 19th century theory into more intricate formulations that, in a sense, extend the notion of Riemann Integral while accommodating a different philosophical argument concerning event evolution.

Although computer experiments suggest the convergence of the sequence $\{ G_{n,n}(z) \}$ for a wide variety of functions $\varphi_{k,n}(z)$, to formulate and prove applicable theorems is no trivial matter. The low hanging fruit is easily plucked, but even rudimental functions $\varphi_{k,n}(z)$ require very delicate investigations.

Example 1 illustrates perhaps the simplest scenario, that of a Riemann Integral. Moving into slightly more complex territory, there is the following :

Theorem 1: Set $g_{k,n}(z) = z + \frac{k}{n^2} f_k(z)$ where $\lim_{k \rightarrow \infty} f_k(z) = c$ uniformly

for all z in a set S . Assume $g_{k,n}(S) \subseteq S$. Then $G_{n,n}(z) \rightarrow z + \frac{c}{2}$ uniformly in S .

Sketch of Proof: Write $G_n = z + \frac{1}{n^2} f_1(z) + \frac{2}{n^2} f_2(G_{1,n}) + \dots + \frac{n}{n^2} f_n(G_{n-1,n})$ and

$T_n = z + \frac{1}{n^2} c + \frac{2}{n^2} c + \dots + \frac{n}{n^2} c \rightarrow z + \frac{c}{2}$. Set $M_k = |f_k(z) - c|$, so that

$$|G_n - T_n| \leq I + II \quad \text{where} \quad I = \frac{1}{n^2} \sum_{k=1}^p k M_k \quad \text{and} \quad II = \frac{1}{n^2} \sum_{k=1}^r (p+k) M_{p+k} \quad \text{with} \quad n = p+r.$$

Choose and fix p so that $M_{p+k} < \frac{\varepsilon}{2}$ for $k \geq 1$. Then $II < \frac{\varepsilon}{2}$ if $n = p+r > R_1 = 3p+1$.

$$\text{Set } \text{Sup}_{1 \leq k \leq p, z \in S} M_k = M, \text{ so that } I < \frac{\varepsilon}{2} \text{ if } r > R_2 = \left\lceil \frac{2Mp^2}{\varepsilon} \right\rceil.$$

Thus $|G_n - T_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ provided $n = p+r > N(\varepsilon) = p + \max\{R_1, R_2\}$. Etc. |

The simplest sequence-generating operators of the form $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$

for non-constant functions $f(z)$ is the subject of

Theorem 2: Set $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$ where $f(z) = \alpha z + \beta$, $\alpha \geq 0$.

Then $G_{n,n}(z) \rightarrow e^{\alpha/2} z + b\beta$ for all complex z .

Sketch of Proof: A little algebra gives

$$G_{n,n}(z) = z \prod_{k=1}^n \left(1 + \frac{k}{n^2} \alpha \right) + \frac{\beta}{n^2} \left[\sum_{k=1}^{n-1} k \prod_{t=k+1}^n \left(1 + \frac{t}{n^2} \alpha \right) + \frac{1}{n} \right]$$

$1 \leq P_n(\alpha) \equiv \prod_{k=1}^n \left(1 + \frac{k}{n^2} \alpha \right) \rightarrow e^{\alpha/2}$. For $0 \leq \alpha < 3$, P_n is monotonic decreasing, and for $3 < \alpha$,

P_n is monotonic increasing.

And $S_n(\alpha) = \frac{1}{n^2} \sum_{k=1}^{n-1} k \prod_{t=k+1}^n \left(1 + \frac{t}{n^2} \alpha \right) \leq (e^\alpha + 1) \cdot \frac{1}{n^2} \sum_{k=1}^n k \leq M$. Thus the monotonically increasing sequence $\{S_n(\alpha)\}$ converges. |

Theorem 2.1 : Set $g_{k,n}(z) = z + \frac{1}{n} f(z)$ with $f(z) = \alpha z + \beta$, $\alpha \geq 0$.

Then $G_{n,n}(z) \rightarrow e^\alpha \left(z + \frac{\beta}{\alpha} \right) - \frac{\beta}{\alpha}$ as $n \rightarrow \infty$

Proof: It is easily verified that $G_{n,n}(z) = z \left(1 + \frac{\alpha}{n} \right)^n + \frac{\beta}{n} \left\{ 1 + \left(1 + \frac{\alpha}{n} \right) + \left(1 + \frac{\alpha}{n} \right)^2 + \dots + \left(1 + \frac{\alpha}{n} \right)^{n-1} \right\}$,

from which the conclusion follows. |

The next result, although of very limited applicability, demonstrates a technical approach to this subject.

Theorem 3: Consider functions $f(x)$ positive, bounded and increasing for positive values of x . Furthermore, assume for a fixed value x_0 that, in a neighborhood of this point,

$$f(x_1) - f(x_2) \geq C(x_0)(x_1 - x_2), \quad x_1 > x_2, \quad C > 4 \quad \text{Set } g_{k,n}(z) = z + \frac{k}{n^2} f(z) \quad \text{and}$$

$G_{k,n}(z) = g_{k,n} \circ g_{k-1,n} \circ \dots \circ g_{1,n}(z)$, and let $z = x$, positive. Then the sequence $\{G_{n,n}(x_0)\}$ converges provided the following condition holds *approximately* for large values of n :

$$f(G_{n,n+1}(x_0)) < 2(G_{n,n+1}(x_0) - x_0) + \frac{1}{4} C(x_0) f(x_0). \quad \text{That is to say, in a}$$

neighborhood* of x_0 ,

$$f(x) < 2(x - x_0) + \frac{1}{4} C(x_0) f(x_0).$$

*It is entirely possible that $G_{n,n+1}$ lies outside that interval, nullifying the theorem.

Argument: For an expansion about x_0 set $G_{k,n} = G_{k,n}(x_0)$. It is easily seen that for a

fixed value of k , $\{G_{k,n}\}$ decreases monotonically to x_0 . Write

$$\begin{aligned} G_{n,n+1} - x_0 &= \left(\frac{n}{n+1}\right)^2 \left[\frac{1}{n^2} f(x_0) + \frac{2}{n^2} f(G_{1,n+1}) + \cdots + \frac{n}{n^2} f(G_{n-1,n+1}) \right] \\ &\leq \left(\frac{n}{n+1}\right)^2 \left[\frac{1}{n^2} f(x_0) + \frac{2}{n^2} (f(G_{1,n}) - \varepsilon_{1,n}) + \cdots + \frac{n}{n^2} (f(G_{n-1,n}) - \varepsilon_{n-1,n}) \right] \end{aligned}$$

Where $f(G_{k,n+1}) + \varepsilon_{k,n} \leq f(G_{k,n})$. For the character of $\varepsilon_{k,n}$ we employ the inequality

$$f(G_{k,n}) - f(G_{k,n+1}) \geq C(x_0)(G_{k,n} - G_{k,n+1}) \geq \cdots \geq C(x_0)f(x_0)\sigma(n) \frac{k(k+1)}{2} \doteq \varepsilon_{k,n}$$

expansion of the second term and minimization of the functional terms. Here $\sigma(n) = \frac{2n+1}{n^2(n+1)^2}$

Then

$$\begin{aligned} G_{n,n+1} - x_0 &\leq \\ &\left(\frac{n}{n+1}\right)^2 \left[G_{n,n} - x_0 - \frac{1}{n^2} \left(2C(x_0)\sigma(n) \frac{1(1+1)}{2} + 3C(x_0)\sigma(n) \frac{2(2+1)}{2} + \cdots + nC(x_0)\sigma(n) \frac{(n-1)n}{2} \right) \right] \\ &\leq \left(\frac{n}{n+1}\right)^2 \left[G_{n,n} - x_0 - \frac{C(x_0)f(x_0)}{2n^2} \cdot \sum_{k=1}^n k(k+1)^2 \right] \\ &= \left(\frac{n}{n+1}\right)^2 \left[G_{n,n} - x_0 - \frac{C(x_0)f(x_0)}{2n^2} \cdot \frac{2n-1}{n(n+1)} \cdot \frac{3n^2+11n+10}{12} \right] \\ &\approx \left(\frac{n}{n+1}\right)^2 [G_{n,n} - x_0 - \delta_n], \quad \delta_n = \frac{C(x_0)f(x_0)}{4n} \end{aligned}$$

So that, approximately,

$$\begin{aligned} G_{n,n} &\geq \left(\frac{n+1}{n}\right)^2 (G_{n,n+1} - x_0) + x_0 + \delta_n \\ &= \left(\frac{n+1}{n}\right)^2 \left[G_{n+1,n+1} - \frac{1}{n+1} f(G_{n,n+1}) - x_0 \right] + x_0 + \delta_n \end{aligned}$$

which gives

$$G_{n,n} - G_{n+1,n+1} \geq \frac{2n+1}{n^2} G_{n+1,n+1} - \frac{n+1}{n^2} f(G_{n,n+1}) - \frac{2n+1}{n^2} x_0 + \delta_n \geq 0$$

\Leftrightarrow

$$G_{n+1,n+1} \geq \frac{n+1}{2n+1} f(G_{n,n+1}) + x_0 - \frac{n^2}{2n+1} \delta_n$$

Writing $G_{n+1,n+1} = G_{n,n+1} + \frac{1}{n+1} f(G_{n,n+1})$ one obtains

$$f(G_{n,n+1}) \leq \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \left[G_{n,n+1} - x_0 + \frac{C(x_0)f(x_0)}{4n} \cdot \frac{n^2}{2n+1} \right],$$

from which the conclusion follows. |

Comment: The hypotheses require

$$C(x_0)(x - x_0) \leq f(x) - f(x_0) \leq 2(x - x_0) + \left(\frac{C(x_0)}{4} - 1\right) f(x_0) \quad \text{or}$$

$$(x - x_0) \leq \frac{1}{4} \left(\frac{C(x_0) - 4}{C(x_0) - 2} \right) f(x_0)$$

The author welcomes better results than this!

Corollary: If $\{G_{n,n}(z)\}$ converges over an interval on the positive real axis and the sequence of analytic functions $\{G_{n,n}(z)\}$ is uniformly bounded over a simply connected set S containing the interval, then $\{G_{n,n}(z)\}$ converges uniformly on every compact subset of S.

Proof: Stieltjes-Vitali Theorem

An Interesting Observation: Set $F_{k,n}(z) = g_{k,n} \circ g_{k+1,n} \circ \dots \circ g_{n,n}(z)$, an *Inner Composition*.

Set $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$. It is easy to see, assuming $f(z) \equiv C$, that

$$\lim_{n \rightarrow \infty} G_{n,n}(z) = \lim_{n \rightarrow \infty} F_{1,n}(z) = z + C \int_0^1 t dt = z + \frac{C}{2}$$

However, it is not so obvious that, in fact, $\lim_{n \rightarrow \infty} G_{n,n}(z) \approx \lim_{n \rightarrow \infty} F_{1,n}(z) \Rightarrow G(z) \approx F(z)$ for more general, non-constant functions $f(z)$.

The Associated Integral: In each example or theorem above the integral associated with the expansion is $\int_0^1 t dt$. However, virtually any proper integral on $[0,1]$ may be used in this context.

Example 3: Set $g_{k,n}(z) = z + \varphi_{k,n}(z) = z + \frac{1}{n+k} f(z)$, $f(z) = z^2$. Here the associated integral is

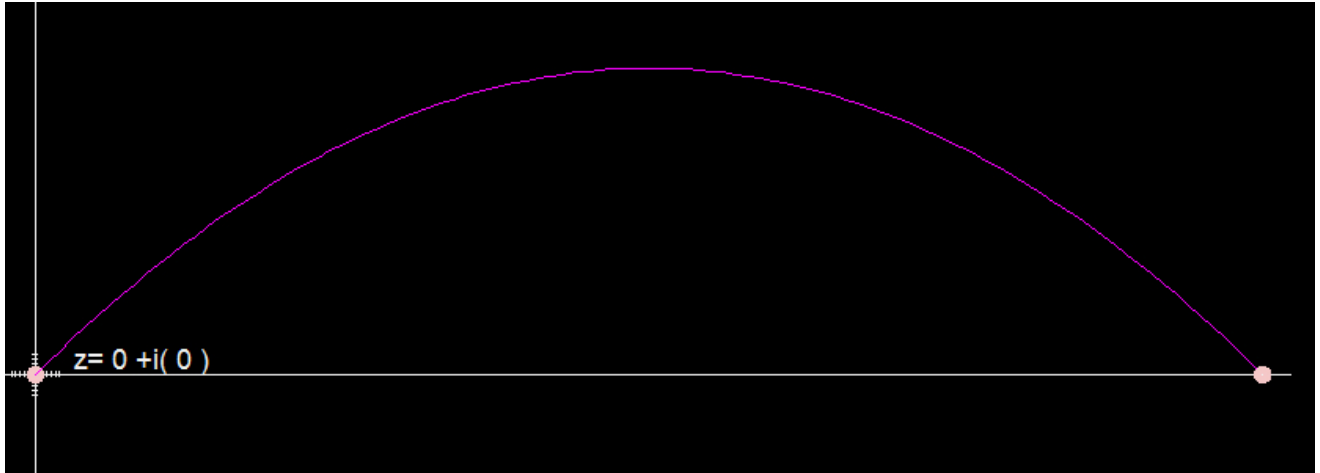
$$\int_0^1 \frac{1}{t+1} dt = \ln 2. \text{ Thus } G_{n,n}(z) = z + \frac{1}{n+1} f(z) + \frac{1}{n+2} f(G_{1,n}) + \dots + \frac{1}{n+n} f(G_{n-1,n}), \text{ and}$$

for instance, $G(.5 + .5i) = .28045... + .91394... i$. Also, $F(.5 + .5i) = .28045... + .91394... i$.

Zeno's Arrow: Standard calculus provides the position of a projectile launched at ground level at an optimum angle of 45 degrees with an initial velocity of V_0 , ignoring air resistance. It can be shown that the corresponding evolution generating functions are

$$g_{k,n}(z) = z + \frac{2v^2}{gn} \left[1 + i \left(1 + \frac{1-2k}{n} \right) \right],$$

where $v = \frac{V_0}{\sqrt{2}} = \frac{g}{2} \eta$ and g = acceleration due to gravity.



$$V_0 = 100, n = 50$$

$$z = 312.5 + i(0)$$

The arrow travels over the time interval $[0, \eta]$, which is divided into subintervals $\left\{ \left[\frac{\eta(k-1)}{n}, \frac{\eta k}{n} \right] \right\}$.

Flight begins at $z=0$ and ends at $x+iy = \frac{V_0^2}{g}$. That is to say, $G_{n,n}(0) \rightarrow \frac{V_0^2}{g}$.

This is a somewhat trivial example of the theory described above, since the second term does not involve the variable z .

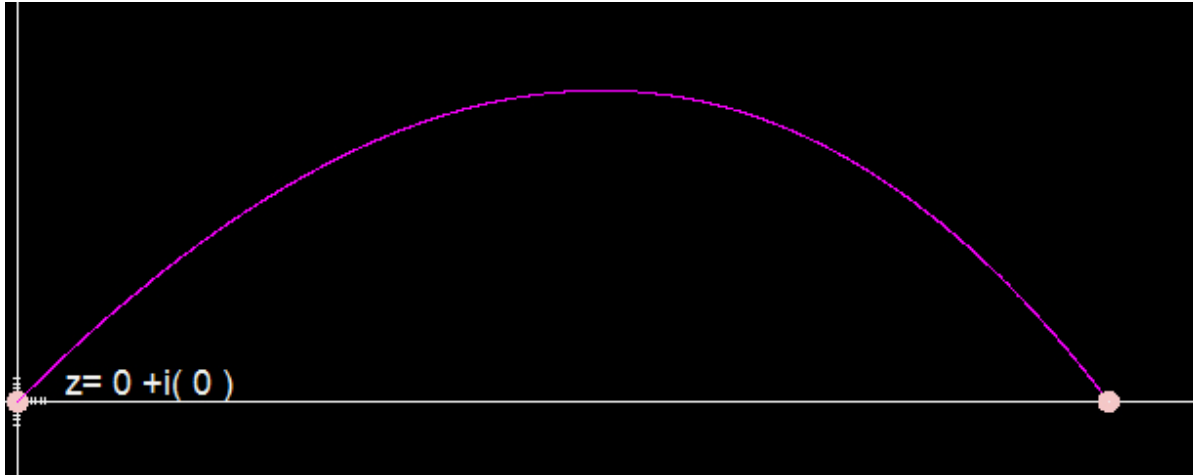
If the force exerted by air resistance is proportional to the speed of the projectile the resulting function looks a bit different. The angle of launch is kept more general here:

Suppose $Force_{air} = \rho V(t)$. Assume m = the mass of the projectile. Now, assume the time interval from launch to impact at ground level is $[0, \eta]$. Divide this interval into subintervals

$\left\{ \left[\frac{\eta k}{n}, \frac{\eta(k+1)}{n} \right] \right\}$. Set $v_{k,n} = e^{-\frac{\rho \eta k}{m n}} \left(e^{\frac{\rho \eta}{m n}} - 1 \right)$. Then the generating evolution functions are

$$g_{k,n}(z) = z + \frac{m}{\rho} \left[v_{k,n} V_0 \cos \theta + i \left(v_{k,n} \left(V_0 \sin \theta + \frac{mg}{\rho} \right) - \frac{\eta g}{n} \right) \right],$$

where $v_{k,n} \rightarrow 0$ as $n \rightarrow \infty$ for $1 \leq k \leq n$.



$$V_0 = 100, n = 2000, \theta = \frac{\pi}{4}$$

$$z = 239.47 + i(0)$$

Is the motion at an “Instant” actually 0? Consider the simple case where the time interval is

$[0,1]$ and $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$, with f bounded. Suppose $t_0 \in (0,1)$.

Then there exists a sequence of intervals $\left\{ \left[\frac{k}{n}, \frac{k+1}{n} \right] \right\}_{n \rightarrow \infty}$ such that

$\frac{k}{n} < t_0 < \frac{k+1}{n}$ and the intervals collapse to the point $t_0 \in (0,1)$. On these intervals, given $\varepsilon > 0$,

$$|g_{k,n}(z) - z| < \varepsilon \text{ for } n \text{ sufficiently large.}$$

What about Continuity? In the context of Zeno’s Arrow or similar motion of a point through space continuity essentially means that, for a small increment of time on the time axis, there is observed a similar small increment of motion of the point or projectile. That is to say

$$|G_{k,n}(z) - G_{k-1,n}(z)| = |g_{k,n}(G_{k-1,n}(z)) - G_{k-1,n}(z)| < \varepsilon \text{ for sufficiently large values of } n.$$

However, assuming f is uniformly bounded by M over a set S , and $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$,

$$|g_{k,n}(G_{k-1,n}(z)) - G_{k-1,n}(z)| \leq \frac{k}{n^2} |f(z)| \leq \frac{1}{n} \cdot \frac{k}{n} M \leq \frac{M}{n} < \varepsilon \quad \text{if } n > \frac{M}{\varepsilon}.$$

Clearly, the condition is satisfied uniformly over the set S . A similar argument suffices for the evolution function describing Zeno's arrow.

Fixed Points: Any discussion of iterative processes should include fixed points. The famous *Contraction Principle* for complex functions is:

Theorem 4 (Henrici [2], 1974) Let f be analytic in a simply-connected region S and continuous on the closure S' of S . Suppose $f(S')$ is a bounded set contained in S . Then $f^n(z) = f \circ f \circ \dots \circ f(z) \rightarrow \alpha$, the *unique attractive fixed point* of f in S , for all z in S' .

Which is generalized by:

Theorem 5 (Lorentzen [3], 1990) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each $f_n(D) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$ converges uniformly in D to a constant function $F(z) = \lambda$.

Theorem 6 (Gill [4], 1991) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D . Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ for all n . Define $G_n(z) = g_n \circ g_{n-1} \circ \dots \circ g_1(z)$. Then $G_n(z) \rightarrow \alpha$ uniformly on the closure of D *if and only if* the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to the number α .

Unfortunately, the evolution functions described in this note are not contractive. However,

it is easily seen that $\varphi_{k,n}(\alpha) = 0 \Rightarrow g_{k,n}(\alpha) = \alpha$.

Theorem 7 Let $g_{k,n}(z) = z + \frac{k}{n^2} f(z)$, $z \in S \Rightarrow g_{k,n}(z) \in S$ Suppose $f(z)$ is analytic on S and that it

satisfies a Lipschitz Condition: $|f(z_1) - f(z_2)| \leq \rho |z_1 - z_2|$ and also $f(\alpha) = 0$.

Then α is a fixed point of $g_{k,n}(z)$ and $|G_{n,n}(z) - \alpha| \leq |z - \alpha| \cdot \prod_{k=1}^n \left(1 + \frac{k}{n^2} \rho\right) \leq \eta(\rho) \cdot |z - \alpha|$

and, if $\lim_{n \rightarrow \infty} G_{n,n}(z) = G(z)$ exists, then $|G'(\alpha)| \leq \eta(\rho)$

Sketch of Proof: The results follow easily from $|g_{k,n}(z) - \alpha| \leq |z - \alpha| + \frac{k}{n^2} \rho \cdot |f(z) - f(\alpha)|$.

Some values for $\eta(\rho) \approx e^{\rho/2}$ are: $\eta(\frac{1}{4}) \approx 1.133$, $\eta(\frac{1}{2}) \approx 1.284$, $\eta(1) \approx 1.648$, $\eta(2) = e$, $\eta(3) \approx 4.481$, $\eta(4) = \infty$

An Integral function arising from these ideas?

Start with $g_{k,n}(z) \equiv z + \frac{1}{n} f(z)$ with $f(z)$ analytic on a domain S , and $z \in S \Rightarrow g_{k,n}(z) \in S$.

Then we have $G_{n,n}(z) = z + \frac{1}{n} f(z) + \frac{1}{n} f(G_{1,n}(z)) + \frac{1}{n} f(G_{2,n}(z)) + \dots + \frac{1}{n} f(G_{n-1,n}(z))$.

Now, imagine a function

$$(1) \quad \boxed{\varphi(z, t), t \in [0, 1] \quad \text{and} \quad \varphi\left(z, \frac{k}{n}\right) \equiv f\left(G_{k-1,n}(z)\right), \text{ with } \int_0^1 \varphi(z, t) dt \text{ defined}}$$

$$\text{Set } \Phi_n(z) = G_{n,n}(z) - z = \frac{1}{n} \varphi\left(z, \frac{1}{n}\right) + \frac{1}{n} \varphi\left(z, \frac{2}{n}\right) + \frac{1}{n} \varphi\left(z, \frac{3}{n}\right) + \dots + \frac{1}{n} \varphi\left(z, \frac{n}{n}\right).$$

Then $\Phi_n(z) \rightarrow \int_0^1 \varphi(z, t) dt \equiv F(z)$, by the definition of the Riemann Integral.

Becoming more specific, let $S = (|z| < R)$ and $S_1 = (|z| < R_1)$ with $R_1 < R$.

Now define $R_2 = \frac{R_1 + R}{2}$ and choose $z \in S_2 = (\{|z| < R_2\})$. Assume $f(S) \subset \overline{S_1}$.

$$\text{Then } |G_{k,n}(z)| < R, \text{ and each } \left| \varphi\left(z, \frac{k}{n}\right) \right| = |f(G_{k-1,n}(z))| < R_1.$$

Since $\{\Phi_n(z)\}$ converges (and contracts) the fixed points $\{\alpha_n\}$ of $\{\Phi_n(z)\}$ converge: $\alpha_n \rightarrow \alpha$.

Define $F_n(z) = \Phi_n \circ \Phi_{n-1} \circ \dots \circ \Phi_1(z)$. Theorem 6 implies $F_n(z) \rightarrow \alpha = \int_0^1 \varphi(\alpha, t) dt$ if and only if

the sequence of fixed points of $\{\Phi_n(z)\}$ converges to that limit.

Example 4: Somewhat trivial, but is a rare case when the closed form of $\varphi(z, t)$ can be approximated.

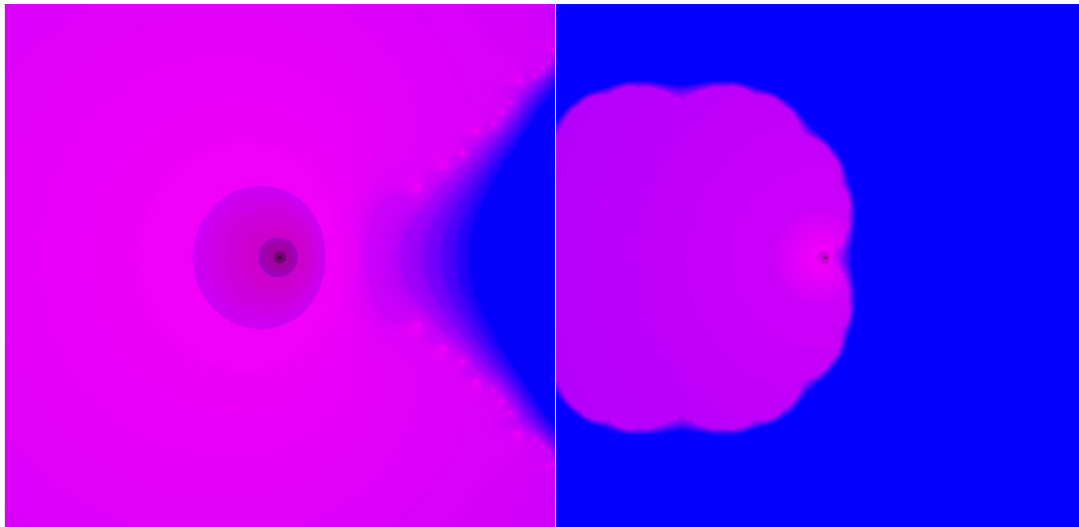
Set $g_n(z) = z - \frac{1}{n} f(z)$, $f(z) = \frac{1}{4} z$. $R_1 = \frac{1}{4}$, $R_2 = \frac{1}{2}$, $R = 1$.

Then $G_{k-1,n}(z) = z \left(1 - \frac{1}{4n}\right)^k = z \left(\left(1 - \frac{1}{4n}\right)^{4n} \right)^{\frac{1}{4} \frac{k}{n}} \approx z e^{-\frac{1}{4} \frac{k}{n}}$ for large values of n ,

with $|G_{k-1,n}(z)| < R_2$.

Thus $\varphi\left(z, \frac{k}{n}\right) = f(G_{k-1,n}(z)) \approx \frac{1}{4} z e^{-\frac{1}{4} \frac{k}{n}} \Rightarrow \varphi(z, t) \approx \frac{1}{4} z e^{-\frac{1}{4} t}$,

and $\int_0^1 \varphi(z, t) dt \approx z(1 - \sqrt{e}) = z \Leftrightarrow z = 0 = \alpha$.



Convergence behavior of $G_{n,n}(z)$ for $g_{k,n}(z) = z + \frac{k}{n^2} \cdot z^2$ on $[-6,6]$ and $[-60,60]$. $N < 10$

Very dark means $|G_{n,n}(z) - z|$ is very small, tapering out to blue, representing either extremely high values or, more likely, divergence. Convergence theory is problematical, but *boundedness* seen above is easy:

Theorem 4: Consider $g_{k,n}(z) = z + \frac{k}{n^2} \cdot f(z)$, $f(0) = 0$, $|f(z)| \leq R$ for $|z| \leq R$.

Then $|G_{n,n}(z)| < R$, $\forall n$ if $z \in S = \left(|z| \leq \frac{R}{2} \right)$.

Sketch of proof:

Schwarz's Lemma implies $|f(z)| \leq |z|$ for $|z| \leq R$.

Hence $|G_{1,n}(z)| \leq |z| + \frac{1}{n^2} |f(z)| \leq \frac{R}{2} \left(1 + \frac{1}{n^2} \right)$ if $z \in S = \left(|z| \leq \frac{R}{2} \right)$

And ... $|G_{n,n}(z)| \leq \frac{R}{2} \prod_{k=1}^n \left(1 + \frac{k}{n^2} \right) < R$ for $z \in S$. |

And, a little wider scope . . .

Theorem 5: Suppose $|z| < R \Rightarrow |f(z)| < M$ where $M < 2R$. Choose $\varepsilon > 0$ such that

$$R_0 = R - M \left(\frac{1}{2} + \varepsilon \right) > 0. \text{ Then } N = N(\varepsilon) = \left\lceil \frac{1}{2\varepsilon} \right\rceil + 1 \text{ and}$$

$$|z| < R_0 \Rightarrow |G_{k,n}(z)| < R \text{ for } n > N.$$

Sketch of proof: $n > \frac{1}{2\varepsilon} \Rightarrow R_0 + \left(\frac{1}{2} + \frac{1}{2n} \right) M < R_0 + \left(\frac{1}{2} + \varepsilon \right) M < R$. Thus

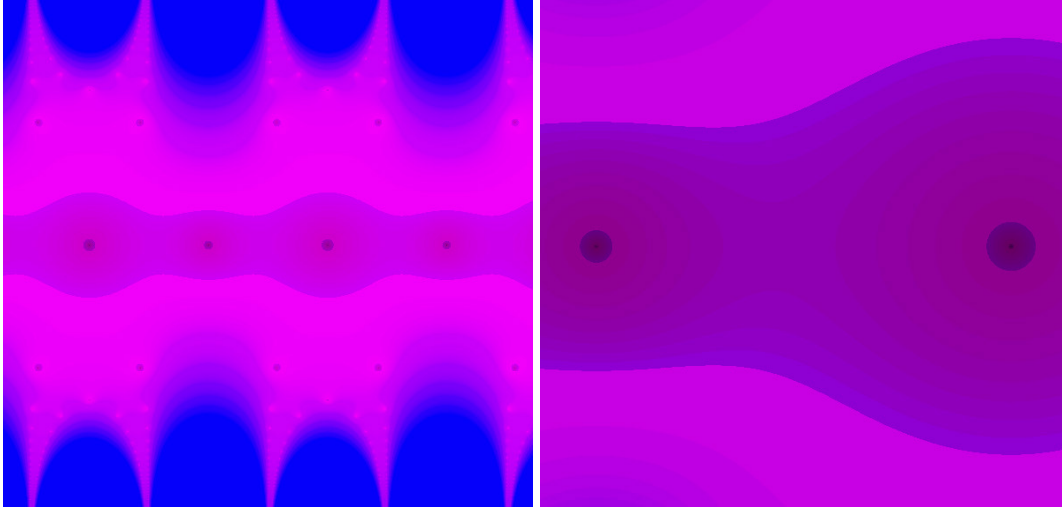
$$|G_{1,n}(z)| \leq |z| + \frac{1}{n^2} |f(z)| < R_0 + \frac{1}{n^2} M \leq R_0 + M \sum_{k=1}^n \frac{k}{n^2} = R_0 + \left(\frac{1}{2} + \frac{1}{2n} \right) M < R,$$

$$|G_{2,n}(z)| \leq |z| + \frac{1}{n^2} |f(z)| + \frac{2}{n^2} |f(G_{1,n}(z))| < R_0 + \left(\frac{1}{n^2} + \frac{2}{n^2} \right) M \leq R_0 + M \sum_{k=1}^n \frac{k}{n^2} = R_0 + \left(\frac{1}{2} + \frac{1}{2n} \right) M < R,$$

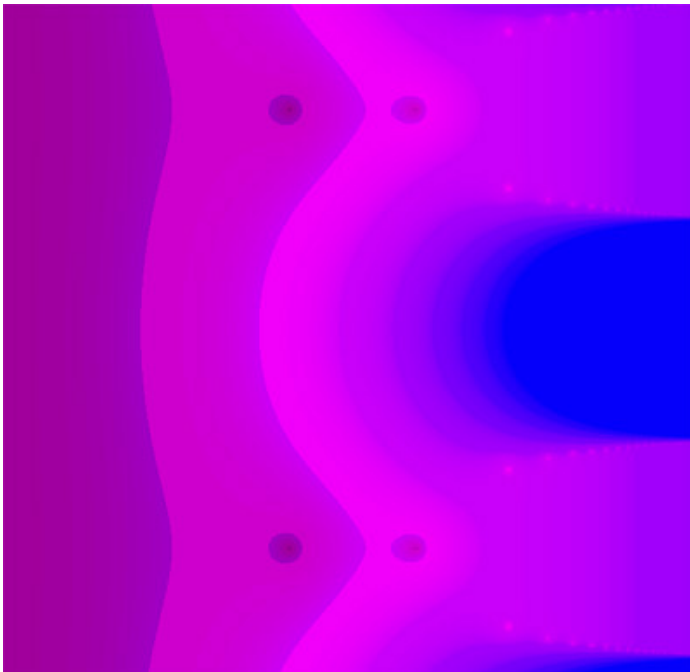
Etc. |

Example: $|z| < R = 1 \Rightarrow \left| \frac{e^z}{10} \right| < .28 = M$. Hence $\varepsilon < 3.08$. Choose $\varepsilon = .10 \Rightarrow N = 6$.

Then $R_0 \approx .83$. Thus $|z| < .83 \Rightarrow |G_{k,n}(z)| < 1$ for $n > 6$.



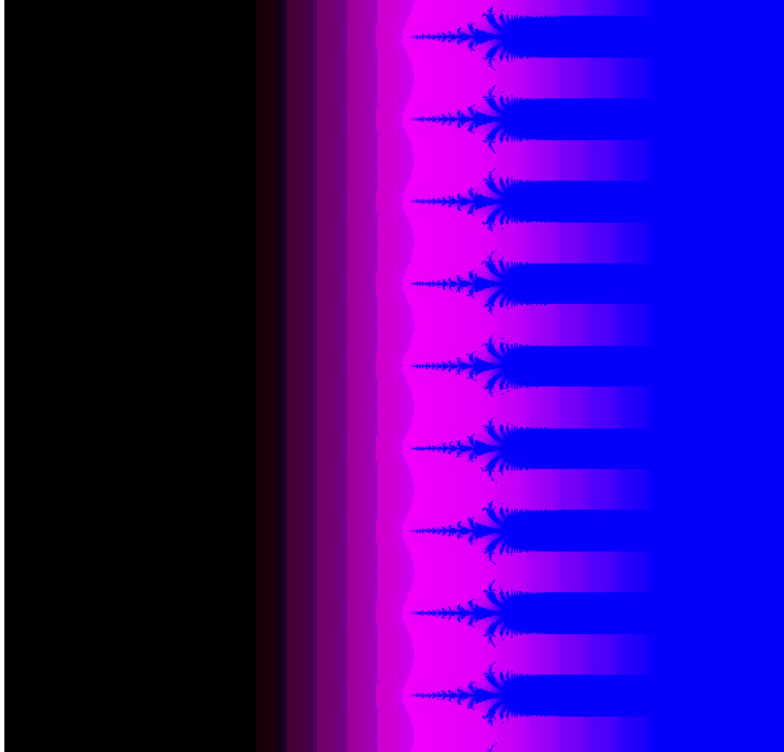
Behavior of $|G_{n,n}(z) - z|$ for $g_{k,n}(z) = z + \frac{k}{n^2} \cdot \cos(z)$ on $[-7,7]$ and $[-2,2]$. $N < 5$



Behavior of $|G_{n,n}(z) - z|$ for $g_{k,n}(z) = z + \frac{1}{n} e^{\left(\frac{z \cdot k}{2^n}\right)}$ on $[-15,25]$, $N=3$



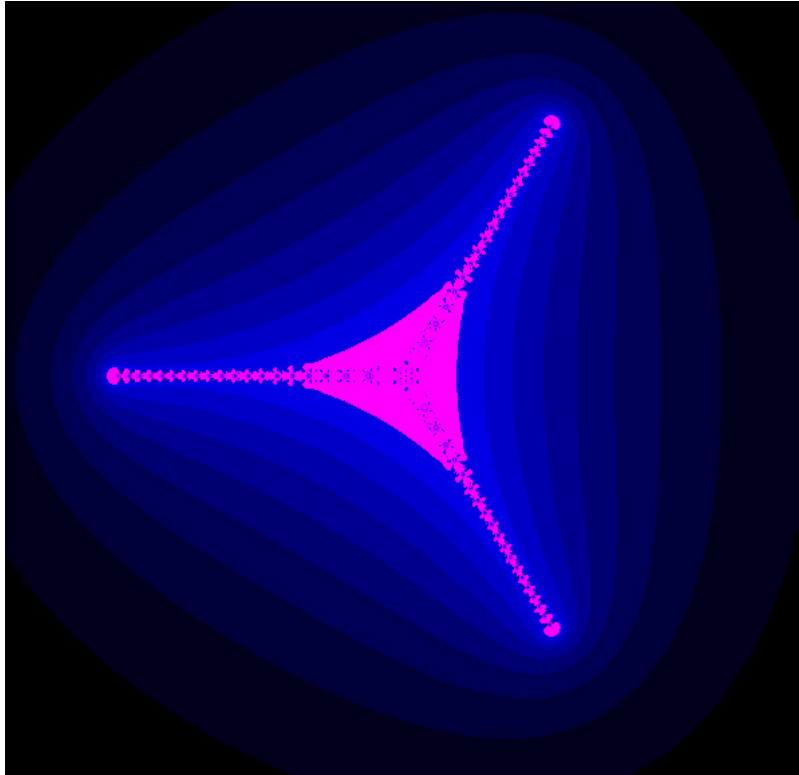
The complexity of convergence: $|G_{n,n}(z) - z|$ for $g_{k,n}(z) = z + \frac{1}{n} e^{\left(\frac{z \cdot k}{2n}\right)}$ on $[-10, 190]$, $N=30$. Here the process of iteration has been altered to replace the absolute value at each stage by a uniform constant when the absolute value is extremely high: the blue fractal “flowers” (Julia Set)



Example: $|G_{n,n}(z) - z|$ for $g_{k,n}(z) = z + \frac{k}{n^2} e^z$ and $\text{Re}(z) < -R, R > 0$. On $[-30, 30]$ for $n=15$.

A little complex algebra produces $|G_{k,n}(z)| \leq |z| + e^{\text{Re}(z)} \cdot \frac{1}{n^2} \sum_{j=1}^k j \leq |z| + e^{\text{Re}(z)}$,

seen graphically as the black area on the above picture.



Example: $g_{k,n}(z) = z + \frac{k}{n^2} \cdot \frac{1}{z^2}$. $[-1.2, 1.2]$, $n=20$. For $R > 1$, the following is not difficult to show:

$|z| > R + \frac{1}{R^2} \Rightarrow |G_{k,n}(z) - z| < \frac{1}{R^2} \cdot \frac{1}{n^2} \sum_{j=1}^k j < \frac{1}{R^2} \cdot \left(\frac{1}{2} + \frac{1}{2n} \right) \leq \frac{1}{R^2}$. Hence the surrounding black region where $|G_{k,n}(z) - z|$ is quite small. The odd, bright limbs at multiples of $\frac{2\pi}{3}$ show points that

move from one branch to another under the iteration.

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