Tannery's Theorem Potpourri . . .

John Gill **www.johngill.net**
7 November 2009

(Theorems, corollaries, etc. without references are due to the author)

Comment : Consider $T_{n,n}(z) = t_{1,n} \circ t_{2,n} \circ \cdots \circ t_{n,n}(z)$ where each function is of the form $t_{k,n}(z) = a_k(n) + z$ and $\lim_{n\to\infty} a_k(n) = a_k$. Then

$$T_{n,n}(0) = a_1(n) + a_2(n) + \cdots + a_n(n) \quad .$$

Tannery's original theorem covered this sort of thing, using uniform convergence properties:

$$\lim_{n\to\infty}[a_1(n) + a_2(n) + \cdots + a_n(n)] = a_1 + a_2 + \cdots$$

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Several examples where TT may or may not apply:

In each instance, $a_k(n) \to a_k \equiv 0$ as $n \to \infty$. Does $\lim_{n\to\infty} \sum_{k=1}^{n} a_k(n) = 0$?

Example 1: $a_k(n) = \frac{k}{n} \Rightarrow \lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n) = \infty$

Example 2: $a_k(n) = \frac{k}{n^2} \Rightarrow \lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n) = \frac{1}{2}$

Example 3: $a_k(n) = \frac{k}{n^3} \Rightarrow \lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n) = a_1 + a_2 + \cdots = 0 + 0 + \cdots = 0$

Example 4: $a_k(n) = \frac{1}{n} f\left(\frac{k}{n}\right)$, $f \in C[0,1] \Rightarrow \lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n) = \int_0^1 f(x)dx$

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In more general settings:

Example 5: $a_k(n)=\begin{cases}0 & \text{if } k<n \\ 1 & \text{if } k=n\end{cases}$ shows that $\lim_{n\to\infty} T_{n,n}(0)=\lim_{n\to\infty}\sum_{k=1}^{n} a_k(n)$ may exist in the absence of the condition $a_n(n)\to 0$.

Example 6: $a_k(n)=\begin{cases}1 & \text{if } k<n \\ 0 & \text{if } k=n\end{cases}$ shows that $a_n(n)\to 0$ does not imply $\lim_{n\to\infty} T_{n,n}(0)=\lim_{n\to\infty}\sum_{k=1}^{n} a_k(n)$ exists (in a finite sense).

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In an effort to extend the scope of TT applied to *forward iteration* (or *inner composition*) the requirement that each function map a domain S into a *compact* subset of S can be weakened a tad. Here, in this elementary theorem, the functions $\{ t_{k,n} \}$ are general in nature.

Theorem 1 : Let S be a simply-connected domain and $\{t_{k,n}\}$, $k \le n$, a sequence of functions analytic in S where $t_{k,n}(S)\subset S$ and $t_{k,n}(z)\to t_k(z)$ for each k Suppose that

(a) $\left|t_{k,n}(z)-t_k(z)\right|<\varepsilon_k(n)\to 0$, as $n\to\infty$ for all z in S, and

(b) $\left|t_k(z_1)-t_k(z_2)\right|<\rho_k\left|z_1-z_2\right|$, $\forall z_1,z_2$ in S.

Then $\left|t_{1,n}\circ t_{2,n}\circ\cdots\circ t_{n,n}(z) - t_1\circ t_2\circ\cdots\circ t_n(z)\right| < \sum_{k=1}^{n}\left(\prod_{j=0}^{k-1}\rho_j\right)\varepsilon_k(n)$

Proof: Set $\Phi_{p,n}=t_{p,n}\circ t_{p+1,n}\circ\cdots\circ t_{n,n}(z)$ and $\Psi_{p,n}=t_p\circ t_{p+1}\circ\cdots\circ t_n(z)$

Then

$$\begin{aligned}
\left|\Phi_{1,n}-\Psi_{1,n}\right| &\le \left|t_{1,n}(\Phi_{2,n})-t_1(\Phi_{2,n})\right| + \left|t_1(\Phi_{2,n})-t_1(\Psi_{2,n})\right| \\
&< \varepsilon_1(n)+\rho_1\left|\Phi_{2,n}-\Psi_{2,n}\right| \\
&< \varepsilon_1(n)+\rho_1\left[\left|t_{2,n}(\Phi_{3,n})-t_2(\Phi_{3,n})\right|+\left|t_2(\Phi_{3,n})-t_2(\Psi_{3,n})\right|\right] \\
&\ \ \vdots \\
&< \sum_{k=1}^{n}\left(\prod_{j=0}^{k-1}\rho_j\right)\varepsilon_k(n)
\end{aligned}$$

Since the values of $\{ \rho_k \}$ are not necessarily less than one, $\{ \Psi_{n,n}(z) \}$ might diverge and $\{ \Phi_{n,n}(z) \}$ simply track that sequence, assuming $t_{k,n} \to t_k$ rapidly enough.

Corollary: Observe the *Tannery Series*

$$S(n) = a_1(n) + a_2(n) + \cdots + a_n(n), \text{ with}$$

$|a_k(n) - a_k| < \varepsilon_{k,n} \le \dfrac{\lambda(k)}{n^\beta}$, where $\lambda(k)$ is a linear function of k and $\beta > 2$.

Then $\lim_{n\to\infty} S(n) = a_1 + a_2 + \cdots$, provided $\sum a_k$ converges.

Tighter conditions are possible, but this simple example shows that TT for series has more latitude.

Alternating Tannery Series

Corollary: Alternating series $S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$ require only that $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$ $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$ for convergence, so it would seem reasonable that a Tannery Series, $S(n) = a_1(n) - a_2(n) + \cdots + (-1)^{n+1} a_n(n)$, in order to converge to the alternating series, should exhibit a fairly rapid convergence of individual terms to those of the series. Theorem 1 is applicable, in that

$|a_k(n) - a_k| < \varepsilon_k(n)$ with $\sum_1^n \varepsilon_k(n) \to 0$ is sufficient to insure the convergence of the alternating Tannery Series:

$$|S(n) - S_n| < \sum_1^n \varepsilon_k(n) \to 0 \ .$$

Example: $S(n) = \dfrac{n^2}{1+n^2} - \dfrac{2n^2}{1+4n^2} + \dfrac{3n^2}{1+9n^2} - \cdots + (-1)^{n+1} \dfrac{n \cdot n^2}{1+n^2 \cdot n^2}$. Here the corresponding alternating series is $S_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + (-1)^{n+1} \dfrac{1}{n}$

We find that

$|a_k(n) - a_k| < \dfrac{1}{n^2}$ so that $|S(n) - S_n| < \dfrac{1}{n} \to 0$.

Tannery Continued Fractions

Corollary: Consider the *Tannery C-Fraction* expansion:

$$\frac{C_1(n)z}{1} + \frac{C_2(n)z}{1} + \cdots + \frac{C_n(n)z}{1}$$

Here $t_{k,n}(w) = \dfrac{C_k(n)z}{1+w}$, with $|w| < r < 1$, $|z| < R$, and $|C_k(n)| < C$.

$|t_{k,n}(w)| < \dfrac{CR}{1-r}$ with $R < \dfrac{r(1\text{-}r)}{C}$ insures $|t_{k,n}(w)| < r$ when $|w| < r$.

From (a) of Theorem 1, $|t_{k,n}(w) - t_k(w)| < \dfrac{R}{1-r}\ |C_k(n) - C_k| < \dfrac{R}{1-r}\ \sigma_k(n)$, and

(b) $|t_{k,n}(w_1) - t_{k,n}(w_2)| < \dfrac{CR}{(1\text{-}r)^2} = \rho$. Therefore

$$\left|t_{1,n} \circ t_{2,n} \circ \cdots \circ t_{n,n}(z) \ - \ t_1 \circ t_2 \circ \cdots \circ t_n(z)\right| \ < \ \frac{R}{1\text{-}r} \sum_{k=1}^{n} \rho^{k-1} \sigma_k(n) .$$

To illustrate, let $r = \dfrac{1}{2}$, $R = \dfrac{1}{4}$, $C=1$, $\sigma_k(n) = \dfrac{k}{n^3}$. Hence $\rho_k \equiv 1$.

This gives, after a few calculations,

$$\left| \frac{C_1(n)z}{1} + \frac{C_2(n)z}{1} + \cdots + \frac{C_n(n)z}{1} \ - \ \frac{C_1 z}{1} + \frac{C_2 z}{1} + \cdots + \frac{C_n z}{1} \right| < \frac{1}{4n}\left(1 + \frac{1}{n}\right) \ \rightarrow \ 0 \text{, so that}$$

$$F_n(z) = \frac{C_1(n)z}{1} + \frac{C_2(n)z}{1} + \cdots + \frac{C_n(n)z}{1} \ \rightarrow \ F(z) \text{ , analytic in } |z| < R .$$

An Integral Test for Tannery Series

The following result is an analogue of the familiar Integral Test for series. It doesn't provide a spectacular new perspective on the subject . . . it's merely a curiosity that could probably be improved:

Theorem 2: Let $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$, and suppose that there exists a non-negative, bounded and differentiable function f(x,t), defined for $x \geq 1$ and $t \geq x$, with $f(k,n) = a_k(n)$, $f_t(x,t) > 0$ and $f_x(x,t) < 0$. Then

$$\lim_{n\to\infty} S(n) \quad \text{exists if and only if} \quad \lim_{n\to\infty} \int_1^n f(x,n)dx \quad \text{exists}$$

Proof: A graphical representation of this structure shows the following:

$$S(n) \leq \int_1^n f(x,n)dx - a_1(n) \quad \text{and} \quad \int_1^n f(x,n)dx + a_n(n) \leq S(n)$$

The hypotheses imply S(n) is a non-decreasing, non-negative sequence.

Other Results for Tannery Series

Theorem 3: Let $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$, and suppose there exists a non-negative, bounded, and differentiable function f(x,t) defined for $x \geq 0$ and $t \geq x$, with $f(k,n) = a_k(n)$. Define $\phi(x) = f(x,x)$. Suppose
$f_x(x,t) \geq 0$, $f_t(x,t) < 0$, and $\phi(x) \to 0$ as $x \to \infty$. Also stipulate $f(1,t) \to 0$ as $t \to \infty$.
Then

$$0 \leq \int_0^{n-1} f(x,n)dx + f(n,n) - S(n) \leq f(n,n) - f(1,n) \to 0 \quad \text{as n becomes infinite.}$$

Proof: The easiest proof involves drawing a simple histogram and studying areas under curves.

Example: $f(x,t) = \frac{x}{t^2}$, then $f_x(x,t) = \frac{1}{t^2} > 0$, $f_t(x,t) = \frac{-2k}{t^3} < 0$ and $\phi(x) = \frac{1}{x} \to 0$.
The theorem shows that $\frac{1}{2}\left(1 - \frac{1}{n^2}\right) + \frac{1}{2} - S(n) \to 0$, or $\lim_{n\to\infty} S(n) = \frac{1}{2}$, as is easily verified by evaluating S(n) directly: $S(n) = \frac{1}{n^2}(1 + 2 + 3 + \cdots + n)$.

Example: A slightly more sophisticated example is the following:

$$S(n) = \frac{1}{1^2+n^2} + \frac{2}{2^2+n^2} + \cdots + \frac{n}{n^2+n^2}$$

Here $f(x,t) = \frac{x}{x^2+t^2}$ and the conditions of the theorem are satisfied for the relevant values of the variables. Thus

$$\frac{1}{2}\text{Ln}\left(2-\frac{2}{n}+\frac{1}{n^2}\right)+\frac{1}{2n} - S(n) \to 0, \quad \text{or } S(n) \to \frac{1}{2}\text{Ln}(2)$$

The convergence is very slow.

Theorem 4: Let $S(n) = a_1(n)+a_2(n)+\cdots+a_n(n)$, and suppose there exists a non-negative, bounded, and differentiable function f(x,t) defined for $x \geq 0$ and $t \geq x$, with $f(k,n) = a_k(n)$. Define $\phi(x) = f(x,x)$. Suppose
$f_x(x,t) < 0$, $f_t(x,t) < 0$, and $\phi(x) \to 0$ as $x \to \infty$, and $f(1,t) \to 0$ as $t \to \infty$
Then

$$0 \leq \int_1^n f(x,n)dx + f(1,n) - S(n) \leq f(1,n) - f(n,n) \to 0 \text{ as } n \text{ becomes}$$

infinite.

Proof: Draw a picture!

Example:

$$S(n) = \frac{n}{0^2+n^2} + \frac{n}{1^2+n^2} + \ldots + \frac{n}{n^2+n^2}$$

The conditions of Theorem 4 are satisfied, giving $\lim_{n\to\infty} S(n) = \pi/4$.