

Extending Tannery's Theorem to Backward Iteration

John Gill

31 October 2009

Theorem 1 (Henrici [1], 1974). Let f be analytic in a simply-connected region S and continuous on the closure S' of S . Suppose $f(S')$ is a bounded set contained in S . Then $f^n(z) = f \circ f \circ \dots \circ f(z) \rightarrow \alpha$, the attractive fixed point of f in S , for all z in S' .

This fundamental result for *contraction mappings* can be extended to an infinite composition of functions arranged as *backward iteration* (or *outer composition*):

Theorem 2: [Gill, [7], 1991) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D . Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ for all n . Define $G_n(z) = g_n \circ g_{n-1} \circ \dots \circ g_1(z)$. If the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to a number α , then $G_n(z) \rightarrow \alpha$ uniformly on the closure of D .

Note: The existence of the $\{\alpha_n\}$ is guaranteed by Theorem 1. That the hypotheses cannot be significantly reduced is shown by the example $g_n(z) = -.5$ for n odd and $g_n(z) = .5$ for n even, in the unit disk ($|z| < 1$). It is not essential that $g_n \rightarrow g$, although that is usually the case. If $g_n \rightarrow g$, then $\alpha_n \rightarrow \alpha$.

Next, consider a sequence of functions $\{g_{k,n}\}$ dependent upon both k and n and defined on a suitable domain D . Define $G_{p,n}(z) = g_{p,n} \circ g_{p-1,n} \circ \dots \circ g_{1,n}(z)$, with $p \leq n$.

Theorem 3: [Gill] Suppose $\{g_{k,n}\}$, with $k \leq n$, is a sequence of functions analytic on a simply-connected domain D and continuous on its closure, with $g_{k,n}(D) \subset \Omega$, a compact subset of D , for all k and n . Let the sequence of fixed points $\{\alpha_{k,n}\}$ of $\{g_{k,n}\}$ converge to α . I.e., $\alpha_{k,n} \rightarrow \alpha$ as both k and $n \rightarrow \infty$, with $k \leq n$. Then $G_{n,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \dots \circ g_{1,n}(z) \rightarrow \alpha$ uniformly on the closure of D .

Comment: When $\lim_{n \rightarrow \infty} g_{k,n}(z) = g_k(z)$, for each value of k , both sequences converge to the limit described in theorem 2.

Proof: The proof is similar to that of theorem 2.

Set $D = \{|z| < 1\}$. Let $\Phi(z) := \frac{z - \alpha}{1 - \bar{\alpha}z}$. Then $\Phi: D \rightarrow D$ is analytic there, with $\Phi(\alpha) = 0$ and $\Phi^{-1}(0) = \alpha$. Set $q_{k,n}(z) = \Phi \circ g_{k,n} \circ \Phi^{-1}(z)$.

Lemma 1: $q_{k,n}(0) \rightarrow 0$ as both $k, n \rightarrow \infty$.

Proof of Lemma 1: This result will follow if $g_{k,n}(\alpha) \rightarrow \alpha$. Write

$$(1) \quad |g_{k,n}(\alpha) - \alpha| \leq |g_{k,n}(\alpha) - g_{k,n}(\alpha_{k,n})| + |\alpha_{k,n} - \alpha|$$

For $\varepsilon > 0$, choose K and N such that $k > K$ and $n > N$ imply each term of the right side of (1) is less than $\frac{\varepsilon}{2}$. This is possible for the first term because the $\{g_{k,n}\}$ are uniformly bounded on D , thus equicontinuous there. Hence $q_{k,n}(0) \rightarrow 0$ as $k, n \rightarrow \infty$.

Now, the existence of the compact set Ω implies $|g_{k,n}(z)| \leq \mu < 1$ for all z in D . Thus

$$(2) \quad \text{Sup}_{k,n} (\text{Sup}_{|z|<1} |q_{k,n}(z)|) = \rho < 1$$

exists.

Since $q_{k,n}(0) \rightarrow 0$ as $k, n \rightarrow \infty$, there exists a sequence $\{\varepsilon_{k,n}\}$ such that $0 \leq \varepsilon_{k,n} \rightarrow 0$ as $k, n \rightarrow \infty$, and $|q_{k,n}(0)| \leq \varepsilon_{k,n}$ for all $k > K$ and $n > N$. (E.g., set $\varepsilon_{k,n} = \text{Sup}_{k > K, n > N} |q_{k,n}(0)|$)

Set $H_{k,n}(z) = \frac{q_{k,n}(z)}{\rho}$. Then $|H_{k,n}(z)| < 1$ for all $|z| < 1$. An application of Schwartz's Lemma [3] gives

$$|H_{k,n}(z)| \leq \frac{|H_{k,n}(0)| + |z|}{1 + |H_{k,n}(0)| \cdot |z|} \leq |H_{k,n}(0)| + |z|.$$

Therefore

$$(3) \quad |q_{k,n}(z)| \leq |q_{k,n}(0)| + |z|\rho.$$

Next, set $Q_{k,n}(z) = q_{k,n} \circ q_{k-1,n} \circ \dots \circ q_{1,n}(z)$ for all k and n . Then from (2),

$$(4) \quad |Q_{k,n}(z)| < \rho < 1 \text{ for all } k \text{ and } n.$$

Writing $p = n + m$, begin an inductive procedure with an arbitrary but large value of n , with the goal of proving that $|Q_{p,p}(z)| \rightarrow 0$ as p tends to infinity. Employing backward recursion, using (3) and (4):

$$\begin{aligned} |Q_{n+m,n+m}(z)| &= |q_{n+m,n+m}(Q_{n+m-1,n+m}(z))| \leq |q_{n+m,n+m}(0)| + \rho |Q_{n+m-1,n+m}(z)| < \varepsilon_{n,n} + \rho |Q_{n+m-1,n+m}(z)| \\ &\leq \varepsilon_{n,n} + \rho \{|q_{n+m-1,n+m}(0)| + \rho |Q_{n+m-2,n+m}(z)|\} \\ &< \varepsilon_{n,n} + \rho \varepsilon_{n,n} + \rho^2 |Q_{n+m-2,n+m}(z)| \\ &\leq \varepsilon_{n,n}(1 + \rho) + \rho^2 \{|q_{n+m-2,n+m}(0)| + \rho |Q_{n+m-3,n+m}(z)|\} \\ &< \varepsilon_{n,n}(1 + \rho) + \varepsilon_{n,n} \rho^2 + \rho^3 |Q_{n+m-3,n+m}(z)| \\ &\leq \varepsilon_{n,n}(1 + \rho + \rho^2) + \rho^3 \{|q_{n+m-3,n+m}(0)| + \rho |Q_{n+m-4,n+m}(z)|\} \\ &< \varepsilon_{n,n}(1 + \rho + \rho^2 + \rho^3) + \rho^4 |Q_{n+m-4,n+m}(z)| \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &< \frac{\varepsilon_{n,n}}{1 - \rho} + \rho^{m-1} |Q_{n+1,n+m}(z)| < \frac{\varepsilon_{n,n}}{1 - \rho} + \rho^m \end{aligned}$$

Thus, if n and m are large enough (p is large enough) both terms of the last expression can be made as small as one wishes. Hence $|Q_{p,p}(z)| \rightarrow 0$ as p tends to infinity.

It follows immediately that $G_{n,n}(z) \rightarrow \alpha$ for all z in D .

It is a simple matter to extend these results to more general simply-connected domains, D , by using appropriate Riemann Mapping Functions.

Example 1. The modified *fixed-point continued fraction* seen before

$$C_n(\omega) = \frac{\alpha_1(\alpha_1+1)}{1 +} \frac{\alpha_2(\alpha_2+1)}{1 +} \dots \frac{\alpha_n(\alpha_n+1)}{1 + \omega}$$

can be reconfigured to give a modified *reverse fixed-point continued fraction*:

$$G_n(\omega) = \frac{\alpha_n(\alpha_n+1)}{1 +} \frac{\alpha_{n-1}(\alpha_{n-1}+1)}{1 +} \dots \frac{\alpha_1(\alpha_1+1)}{1 + \omega}$$

convergent when $|\alpha_n| < \frac{1}{5}$, $|\omega| < \frac{1}{2}$, and $\lim_{n \rightarrow \infty} \alpha_n = \alpha$. The $\{\alpha_n\}$ are the *attractive fixed points* of the linear fractional transformations $t_k(\omega) = \frac{\alpha_k(\alpha_k+1)}{1+\omega}$.

Thus, one may write $G_n(\omega) = t_n \circ t_{n-1} \circ \dots \circ t_1(\omega) \rightarrow \alpha$, as $n \rightarrow \infty$.

Setting
$$G_{n,n}(\omega) = \frac{\alpha_n(n)(\alpha_n(n)+1)}{1 +} \frac{\alpha_{n-1}(n)(\alpha_{n-1}(n)+1)}{1 +} \dots \frac{\alpha_1(n)(\alpha_1(n)+1)}{1 + \omega},$$

where $\lim_{n \rightarrow \infty} \alpha_k(n) = \alpha_k$ for each k , and $\lim_{k \rightarrow \infty} \alpha_k = \alpha$, we have

$$\lim_{n \rightarrow \infty} G_{n,n}(\omega) = \lim_{n \rightarrow \infty} G_n(\omega) = \alpha.$$

References:

- [1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1 (Wiley, 1974)
- [3] Z. Nehari, *Conformal Mapping* (McGraw-Hill, 1952)
- [7] J. Gill, The Use of the Sequence $F_n(z) = f_n \circ \dots \circ f_1(z)$ in Computing Fixed Points of Continued Fractions, Products, and Series, *Appl. Numer. Math.* 8 (1991) 469-476