

## Extending Tannery's Theorem to Backward Iteration

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**Theorem 1** (Henrici [1], 1974). Let  $f$  be analytic in a simply-connected region  $S$  and continuous on the closure  $S'$  of  $S$ . Suppose  $f(S')$  is a bounded set contained in  $S$ . Then  $f^n(z) = f \circ f \circ \dots \circ f(z) \rightarrow \alpha$ , the attractive fixed point of  $f$  in  $S$ , for all  $z$  in  $S'$ .

This fundamental result for *contraction mappings* can be extended to an infinite composition of functions arranged as *backward iteration* (or *outer composition*):

**Theorem 2:** [Gill, [7], 1991) Let  $\{g_n\}$  be a sequence of functions analytic on a simply-connected domain  $D$  and continuous on the closure of  $D$ . Suppose there exists a compact set  $\Omega \subset D$  such that  $g_n(D) \subset \Omega$  for all  $n$ . Define  $G_n(z) = g_n \circ g_{n-1} \circ \dots \circ g_1(z)$ . If the sequence of *fixed points*  $\{\alpha_n\}$  of the  $\{g_n\}$  in  $\Omega$  converge to a number  $\alpha$ , then  $G_n(z) \rightarrow \alpha$  uniformly on the closure of  $D$ .

**Note:** The existence of the  $\{\alpha_n\}$  is guaranteed by Theorem 1. That the hypotheses cannot be significantly reduced is shown by the example  $g_n(z) = -.5$  for  $n$  odd and  $g_n(z) = .5$  for  $n$  even, in the unit disk ( $|z| < 1$ ). It is not essential that  $g_n \rightarrow g$ , although that is usually the case. If  $g_n \rightarrow g$ , then  $\alpha_n \rightarrow \alpha$ .

Next, consider a sequence of functions  $\{g_{k,n}\}$  dependent upon both  $k$  and  $n$  and defined on a suitable domain  $D$ . Define  $G_{p,n}(z) = g_{p,n} \circ g_{p-1,n} \circ \dots \circ g_{1,n}(z)$ , with  $p \leq n$ .

**Theorem 3:** [Gill] Suppose  $\{g_{k,n}\}$ , with  $k \leq n$ , is a sequence of functions analytic on a simply-connected domain  $D$  and continuous on its closure, with  $g_{k,n}(D) \subset \Omega$ , a compact subset of  $D$ , for all  $k$  and  $n$ . Let the sequence of fixed points  $\{\alpha_{k,n}\}$  of  $\{g_{k,n}\}$  converge to  $\alpha$ . I.e.,  $\alpha_{k,n} \rightarrow \alpha$  as both  $k$  and  $n \rightarrow \infty$ , with  $k \leq n$ . Then  $G_{n,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \dots \circ g_{1,n}(z) \rightarrow \alpha$  uniformly on the closure of  $D$ .

*Comment:* When  $\lim_{n \rightarrow \infty} g_{k,n}(z) = g_k(z)$ , for each value of  $k$ , both sequences converge to the limit described in theorem 2.

*Proof:* The proof is similar to that of theorem 2.

Set  $D = \{|z| < 1\}$ . Let  $\Phi(z) := \frac{z - \alpha}{1 - \bar{\alpha}z}$ . Then  $\Phi: D \rightarrow D$  is analytic there, with  $\Phi(\alpha) = 0$  and  $\Phi^{-1}(0) = \alpha$ . Set  $q_{k,n}(z) = \Phi \circ g_{k,n} \circ \Phi^{-1}(z)$ .

**Lemma 1:**  $q_{k,n}(0) \rightarrow 0$  as both  $k, n \rightarrow \infty$ .

*Proof of Lemma 1:* This result will follow if  $g_{k,n}(\alpha) \rightarrow \alpha$ . Write

$$(1) \quad |g_{k,n}(\alpha) - \alpha| \leq |g_{k,n}(\alpha) - g_{k,n}(\alpha_{k,n})| + |\alpha_{k,n} - \alpha|$$

For  $\varepsilon > 0$ , choose  $K$  and  $N$  such that  $k > K$  and  $n > N$  imply each term of the right side of (1) is less than  $\frac{\varepsilon}{2}$ . This is possible for the first term because the  $\{g_{k,n}\}$  are uniformly bounded on  $D$ , thus equicontinuous there. Hence  $q_{k,n}(0) \rightarrow 0$  as  $k, n \rightarrow \infty$ .

Now, the existence of the compact set  $\Omega$  implies  $|g_{k,n}(z)| \leq \mu < 1$  for all  $z$  in  $D$ . Thus

$$(2) \quad \text{Sup}_{k,n} (\text{Sup}_{|z|<1} |q_{k,n}(z)|) = \rho < 1$$

exists.

Since  $q_{k,n}(0) \rightarrow 0$  as  $k, n \rightarrow \infty$ , there exists a sequence  $\{\varepsilon_{k,n}\}$  such that  $0 \leq \varepsilon_{k,n} \rightarrow 0$  as  $k, n \rightarrow \infty$ , and  $|q_{k,n}(0)| \leq \varepsilon_{k,n}$  for all  $k > K$  and  $n > N$ . (E.g., set  $\varepsilon_{k,n} = \text{Sup}_{k > K, n > N} |q_{k,n}(0)|$ )

Set  $H_{k,n}(z) = \frac{q_{k,n}(z)}{\rho}$ . Then  $|H_{k,n}(z)| < 1$  for all  $|z| < 1$ . An application of Schwartz's Lemma [3] gives

$$|H_{k,n}(z)| \leq \frac{|H_{k,n}(0)| + |z|}{1 + |H_{k,n}(0)| \cdot |z|} \leq |H_{k,n}(0)| + |z|.$$

Therefore

$$(3) \quad |q_{k,n}(z)| \leq |q_{k,n}(0)| + |z|\rho.$$

Next, set  $Q_{k,n}(z) = q_{k,n} \circ q_{k-1,n} \circ \dots \circ q_{1,n}(z)$  for all  $k$  and  $n$ . Then from (2),

$$(4) \quad |Q_{k,n}(z)| < \rho < 1 \text{ for all } k \text{ and } n.$$

Writing  $p = n + m$ , begin an inductive procedure with an arbitrary but large value of  $n$ , with the goal of proving that  $|Q_{p,p}(z)| \rightarrow 0$  as  $p$  tends to infinity. Employing backward recursion, using (3) and (4):

$$\begin{aligned} |Q_{n+m,n+m}(z)| &= |q_{n+m,n+m}(Q_{n+m-1,n+m}(z))| \leq |q_{n+m,n+m}(0)| + \rho |Q_{n+m-1,n+m}(z)| < \varepsilon_{n,n} + \rho |Q_{n+m-1,n+m}(z)| \\ &\leq \varepsilon_{n,n} + \rho \{|q_{n+m-1,n+m}(0)| + \rho |Q_{n+m-2,n+m}(z)|\} \\ &< \varepsilon_{n,n} + \rho \varepsilon_{n,n} + \rho^2 |Q_{n+m-2,n+m}(z)| \\ &\leq \varepsilon_{n,n}(1 + \rho) + \rho^2 \{|q_{n+m-2,n+m}(0)| + \rho |Q_{n+m-3,n+m}(z)|\} \\ &< \varepsilon_{n,n}(1 + \rho) + \varepsilon_{n,n} \rho^2 + \rho^3 |Q_{n+m-3,n+m}(z)| \\ &\leq \varepsilon_{n,n}(1 + \rho + \rho^2) + \rho^3 \{|q_{n+m-3,n+m}(0)| + \rho |Q_{n+m-4,n+m}(z)|\} \\ &< \varepsilon_{n,n}(1 + \rho + \rho^2 + \rho^3) + \rho^4 |Q_{n+m-4,n+m}(z)| \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &< \frac{\varepsilon_{n,n}}{1 - \rho} + \rho^{m-1} |Q_{n+1,n+m}(z)| < \frac{\varepsilon_{n,n}}{1 - \rho} + \rho^m \end{aligned}$$

Thus, if  $n$  and  $m$  are large enough ( $p$  is large enough) both terms of the last expression can be made as small as one wishes. Hence  $|Q_{p,p}(z)| \rightarrow 0$  as  $p$  tends to infinity.

It follows immediately that  $G_{n,n}(z) \rightarrow \alpha$  for all  $z$  in  $D$ .

It is a simple matter to extend these results to more general simply-connected domains,  $D$ , by using appropriate Riemann Mapping Functions.

**Example 1.** The modified *fixed-point continued fraction* seen before

$$C_n(\omega) = \frac{\alpha_1(\alpha_1+1)}{1 +} \frac{\alpha_2(\alpha_2+1)}{1 +} \dots \frac{\alpha_n(\alpha_n+1)}{1 + \omega}$$

can be reconfigured to give a modified *reverse fixed-point continued fraction*:

$$G_n(\omega) = \frac{\alpha_n(\alpha_n+1)}{1 +} \frac{\alpha_{n-1}(\alpha_{n-1}+1)}{1 +} \dots \frac{\alpha_1(\alpha_1+1)}{1 + \omega}$$

convergent when  $|\alpha_n| < \frac{1}{5}$ ,  $|\omega| < \frac{1}{2}$ , and  $\lim_{n \rightarrow \infty} \alpha_n = \alpha$ . The  $\{\alpha_n\}$  are the *attractive fixed points* of the linear fractional transformations  $t_k(\omega) = \frac{\alpha_k(\alpha_k+1)}{1+\omega}$ .

Thus, one may write  $G_n(\omega) = t_n \circ t_{n-1} \circ \dots \circ t_1(\omega) \rightarrow \alpha$ , as  $n \rightarrow \infty$ .

Setting 
$$G_{n,n}(\omega) = \frac{\alpha_n(n)(\alpha_n(n)+1)}{1 +} \frac{\alpha_{n-1}(n)(\alpha_{n-1}(n)+1)}{1 +} \dots \frac{\alpha_1(n)(\alpha_1(n)+1)}{1 + \omega},$$

where  $\lim_{n \rightarrow \infty} \alpha_k(n) = \alpha_k$  for each  $k$ , and  $\lim_{k \rightarrow \infty} \alpha_k = \alpha$ , we have

$$\lim_{n \rightarrow \infty} G_{n,n}(\omega) = \lim_{n \rightarrow \infty} G_n(\omega) = \alpha.$$

## References:

- [1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1 (Wiley, 1974)
- [3] Z. Nehari, *Conformal Mapping* (McGraw-Hill, 1952)
- [7] J. Gill, The Use of the Sequence  $F_n(z) = f_n \circ \dots \circ f_1(z)$  in Computing Fixed Points of Continued Fractions, Products, and Series, *Appl. Numer. Math.* 8 (1991) 469-476