

## Extending Tannery's Theorem to Forward Iteration: A Tannery Transformation

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The following is a fundamental theorem on *contraction maps*:

**Theorem** (Henrici [1], 1974). Let  $f$  be analytic in a simply-connected region  $S$  and continuous on the closure  $S'$  of  $S$ . Suppose  $f(S')$  is a bounded set contained in  $S$ . Then  $f^n(z) = f \circ f \circ \dots \circ f(z) \rightarrow \alpha$ , the *attractive fixed point* of  $f$  in  $S$ , for all  $z$  in  $S'$ .

This result can be extended to *forward iteration* (or *inner composition*) involving a sequence of functions:

**Theorem:**(Lorentzen, [5],1990) Let  $\{f_n\}$  be a sequence of functions analytic on a simply-connected domain  $D$ . Suppose there exists a compact set  $\Omega \subset D$  such that for each  $n$ ,  $f_n(D) \subset \Omega$ . Then  $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$  converges uniformly in  $D$  to a constant function  $F(z) = \lambda$ .

(Note: This result is sometimes called the *Lorentzen-Gill Theorem* since the second author obtained the result in a specific case in previous papers [4], [6])

The concept underlying *Tannery's Theorem* extends easily to this setting:

**Theorem:** (Gill, [8],1992) Let  $\{f_n\}$  be a sequence of functions analytic on a simply-connected domain  $D$ . Suppose there exists a compact set  $\Omega \subset D$  such that for each  $n$ ,  $f_n(D) \subset \Omega$ . Now suppose there exists a sequence of functions analytic on  $D$  and depending upon both  $k$  and  $n$ ,  $\{f_{k,n}\}$ , such that  $f_{k,n}(D) \subset \Omega$  and  $\lim_{n \rightarrow \infty} f_{k,n}(z) = f_k(z)$  uniformly on  $D$  for each  $k$ . Then, with  $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$  and  $F_{p,n}(z) = f_{1,n} \circ f_{2,n} \circ \dots \circ f_{p,n}(z)$ ,

$$F_{n,n}(z) \rightarrow \lambda = \lim_{n \rightarrow \infty} F_n(z) \text{ as } n \rightarrow \infty, \text{ uniformly on } D.$$

*Proof:* The *Lorentzen – Gill Theorem* defines  $\lambda$ . Write  $Z_{p,n} = f_{p+1,n} \circ f_{p+2,n} \circ \dots \circ f_{n,n}(z)$ . Then

$$\begin{aligned} |F_{n,n}(z) - \lambda| &= |F_{p,n}(Z_{p,n}) - \lambda| \\ &\leq |F_{p,n}(Z_{p,n}) - F_p(Z_{p,n})| + |F_p(Z_{p,n}) - \lambda| \end{aligned}$$

For the second term in the inequality, choose and fix  $p$  sufficiently large that  $|F_p(z) - \lambda| < \frac{\epsilon}{2}$  for all  $z$  in

$D$ . For the first term, choose  $n$  sufficiently large to insure  $|F_{p,n}(z) - F_p(z)| < \frac{\epsilon}{2}$  for all  $z$  in  $D$ . This is true since a finite composition of functions of the type described above, converging uniformly on  $D$ , will also converge uniformly on  $D$ .

**A Tannery Transformation:** An existing compositional structure  $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$  may be transformed using these ideas:

**Corollary:** Let  $\{f_n\}$  be a sequence of functions analytic on a simply-connected domain  $D$ . Suppose there exists a compact set  $\Omega \subset D$  such that for each  $n$ ,  $f_n(D) \subset \Omega$ . Now suppose there exists a sequence of functions analytic on  $D$  and depending upon both  $k$  and  $n$ ,  $\{t_{k,n}\}$ , such that  $t_{k,n}(D) \subset \Omega$  and  $\lim_{n \rightarrow \infty} t_{k,n}(z) = z$  uniformly on  $D$ , for each  $k$ .

Then

$$T_n(z) = f_1 \circ t_{1,n} \circ f_2 \circ t_{2,n} \circ \dots \circ f_n \circ t_{n,n}(z) \rightarrow \lambda,$$

where  $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z) \rightarrow \lambda$ .

*Proof:* Set  $g_{k,n}(z) = f_k(t_{k,n}(z))$  and apply the theorem.

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**Example 1.** The modified *fixed-point continued fraction*

$$C_n(\omega) = \frac{\alpha_1(\alpha_1 + 1)}{1 + \omega} \cdot \frac{\alpha_2(\alpha_2 + 1)}{1 + \omega} \cdot \dots \cdot \frac{\alpha_n(\alpha_n + 1)}{1 + \omega}$$

converges under the following stipulations:  $|\alpha_n| < \frac{1}{5}$ ,  $|\omega| < \frac{1}{2}$ . The  $\{\alpha_n\}$  are the

*attractive fixed points* of the linear fractional transformations  $t_k(\omega) = \frac{\alpha_k(\alpha_k + 1)}{1 + \omega}$ .

Thus, one may write  $C_n(\omega) = t_1 \circ t_2 \circ \dots \circ t_n(\omega)$ . (If  $\alpha_n \equiv \alpha$ , then  $\lim_{n \rightarrow \infty} C_n(\omega) = \alpha$ ).

Writing  $C_{n,n}(\omega) = \frac{\alpha_1(n)(\alpha_1(n) + 1)}{1 + \omega} \cdot \frac{\alpha_2(n)(\alpha_2(n) + 1)}{1 + \omega} \cdot \dots \cdot \frac{\alpha_n(n)(\alpha_n(n) + 1)}{1 + \omega}$ , where

$\lim_{n \rightarrow \infty} \alpha_k(n) = \alpha_k$  for each  $k$ , we have  $\lim_{n \rightarrow \infty} C_{n,n}(\omega) = \lim_{n \rightarrow \infty} C_n(\omega)$ .

## References:

- [1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1 (Wiley, 1974)
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- [6] J. Gill, Limit Periodic Iteration, *Appl. Numer. Math.* 4 (1988) 297-308
- [8] J. Gill, A Tannery Transformation of Continued Fractions and Other Expansions, *Comm. Anal. Th. Cont. Fraction* Vol. 1, (1992)