

Extending Tannery's Theorem to Forward Iteration: A Tannery Transformation

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The following is a fundamental theorem on *contraction maps*:

Theorem (Henrici [1], 1974). Let f be analytic in a simply-connected region S and continuous on the closure S' of S . Suppose $f(S')$ is a bounded set contained in S . Then $f^n(z) = f \circ f \circ \dots \circ f(z) \rightarrow \alpha$, the *attractive fixed point* of f in S , for all z in S' .

This result can be extended to *forward iteration* (or *inner composition*) involving a sequence of functions:

Theorem:(Lorentzen, [5],1990) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each n , $f_n(D) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$ converges uniformly in D to a constant function $F(z) = \lambda$.

(Note: This result is sometimes called the *Lorentzen-Gill Theorem* since the second author obtained the result in a specific case in previous papers [4], [6])

The concept underlying *Tannery's Theorem* extends easily to this setting:

Theorem: (Gill, [8],1992) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each n , $f_n(D) \subset \Omega$. Now suppose there exists a sequence of functions analytic on D and depending upon both k and n , $\{f_{k,n}\}$, such that $f_{k,n}(D) \subset \Omega$ and $\lim_{n \rightarrow \infty} f_{k,n}(z) = f_k(z)$ uniformly on D for each k . Then, with $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$ and $F_{p,n}(z) = f_{1,n} \circ f_{2,n} \circ \dots \circ f_{p,n}(z)$,

$$F_{n,n}(z) \rightarrow \lambda = \lim_{n \rightarrow \infty} F_n(z) \text{ as } n \rightarrow \infty, \text{ uniformly on } D.$$

Proof: The *Lorentzen – Gill Theorem* defines λ . Write $Z_{p,n} = f_{p+1,n} \circ f_{p+2,n} \circ \dots \circ f_{n,n}(z)$. Then

$$\begin{aligned} |F_{n,n}(z) - \lambda| &= |F_{p,n}(Z_{p,n}) - \lambda| \\ &\leq |F_{p,n}(Z_{p,n}) - F_p(Z_{p,n})| + |F_p(Z_{p,n}) - \lambda| \end{aligned}$$

For the second term in the inequality, choose and fix p sufficiently large that $|F_p(z) - \lambda| < \frac{\varepsilon}{2}$ for all z in

D . For the first term, choose n sufficiently large to insure $|F_{p,n}(z) - F_p(z)| < \frac{\varepsilon}{2}$ for all z in D . This is true since a finite composition of functions of the type described above, converging uniformly on D , will also converge uniformly on D .

A Tannery Transformation: An existing compositional structure $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z)$ may be transformed using these ideas:

Corollary: Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each n , $f_n(D) \subset \Omega$. Now suppose there exists a sequence of functions analytic on D and depending upon both k and n , $\{t_{k,n}\}$, such that $t_{k,n}(D) \subset \Omega$ and $\lim_{n \rightarrow \infty} t_{k,n}(z) = z$ uniformly on D , for each k .

Then

$$T_n(z) = f_1 \circ t_{1,n} \circ f_2 \circ t_{2,n} \circ \dots \circ f_n \circ t_{n,n}(z) \rightarrow \lambda,$$

where $F_n(z) = f_1 \circ f_2 \circ \dots \circ f_n(z) \rightarrow \lambda$.

Proof: Set $g_{k,n}(z) = f_k(t_{k,n}(z))$ and apply the theorem.

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Example 1. The modified *fixed-point continued fraction*

$$C_n(\omega) = \frac{\alpha_1(\alpha_1 + 1)}{1 + \omega} \cdot \frac{\alpha_2(\alpha_2 + 1)}{1 + \omega} \cdot \dots \cdot \frac{\alpha_n(\alpha_n + 1)}{1 + \omega}$$

converges under the following stipulations: $|\alpha_n| < \frac{1}{5}$, $|\omega| < \frac{1}{2}$. The $\{\alpha_n\}$ are the

attractive fixed points of the linear fractional transformations $t_k(\omega) = \frac{\alpha_k(\alpha_k + 1)}{1 + \omega}$.

Thus, one may write $C_n(\omega) = t_1 \circ t_2 \circ \dots \circ t_n(\omega)$. (If $\alpha_n \equiv \alpha$, then $\lim_{n \rightarrow \infty} C_n(\omega) = \alpha$).

Writing $C_{n,n}(\omega) = \frac{\alpha_1(n)(\alpha_1(n) + 1)}{1 + \omega} \cdot \frac{\alpha_2(n)(\alpha_2(n) + 1)}{1 + \omega} \cdot \dots \cdot \frac{\alpha_n(n)(\alpha_n(n) + 1)}{1 + \omega}$, where

$\lim_{n \rightarrow \infty} \alpha_k(n) = \alpha_k$ for each k , we have $\lim_{n \rightarrow \infty} C_{n,n}(\omega) = \lim_{n \rightarrow \infty} C_n(\omega)$.

References:

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