Generalizations of the Classical Tannery's Theorem

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ABSTRACT: *Inner Composition* of analytic functions ($f_1 \circ f_2 \circ \cdots \circ f_n(z)$) and *Outer Composition* of analytic functions ($f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$) are variations on simple iteration, and their convergence behaviors may reflect that of simple iteration of a *contraction mapping* described by Henrici [3]. Investigations of the more complicated structures $f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(z)$ and $f_{n,n} \circ f_{n-1,n} \circ \cdots \circ f_{1,n}(z)$ lead to extensions of the classical Tannery's Theorem [1]. A variety of examples and original minor theorems related to the topic are presented. The paper is devised in the spirit of elementary classical analysis and much is accessible to serious undergraduate majors; there is little reference to modern, or "soft" analysis. [*AMS Subject Classifications 40A30, primary,30E99, secondary. October 2010*]

1. **Preliminaries**:

Tannery's Theorem [1] provides sufficient conditions on the series-like expression S(n) = $a_1(n) + a_2(n) + \cdots + a_n(n)$ that it converge to the limit of the series $a_1 + a_2 + \cdots$, when $\lim_{n\to\infty} a_k(n) = a_k$ for each k. In fact, the original theorem provided this result for a more general series-like expansion , S(p,n) = $a_1(n) + a_2(n) + \cdots + a_p(n)$, where it is understood that p tends steadily to infinity with n. <u>In this and subsequent notes p will be taken to be n</u>.

Tannery's Theorem (series): Suppose that $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$, where $\lim_{n\to\infty} a_k(n) = a_k$ for each k. Furthermore, assume $|a_k(n)| \le M_k$ with $\sum M_k < \infty$.

Then $\lim_{n\to\infty} S(n) = a_1 + a_2 + \cdots$, convergent.

Proof: (sketch) Write

$$\begin{aligned}&\left|a_1(n) + a_2(n) + \cdots + a_n(n) - (a_1 + a_2 + \cdots + a_n)\right| \\ &\le \sum_{k=1}^{p} |a_k(n) - a_k| + \sum_{k=p+1}^{n} |a_k(n)| + \sum_{k=p+1}^{n} |a_k|\end{aligned} ,$$

Etc. ǁ

Setting $f_{k,n}(z) = a_k(n) + z$, then one may write

$$S(n) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(0), \text{ or } \quad S(n) = f_{n,n} \circ f_{n-1,n} \circ \cdots \circ f_{1,n}(0) .$$

Comment: The hypotheses can be weakened – see Tannery's Theorem Potpourri.

The classical Tannery theory can easily be extended to infinite products:

Tannery's Theorem (products): Suppose that $P(n) = \prod_{k=1}^{n}(1 + a_k(n))$. If $\lim_{n\to\infty} a_k(n) = a_k$, and $|a_k(n)| \le M_k$ with $\sum M_k < \infty$, then $\lim_{n\to\infty} P(n) = \prod_{k=1}^{\infty}(1 + a_k)$.

Proof: (sketch) $\prod\left(1 + a_k(n)\right) = e^{\sum \ln(1+a_k(n))}$, $|\ln(1+z)| < \frac{3}{2}|z|$ if $|z| < \frac{1}{2}$
Apply Tannery's Theorem for series . . . ||

However, the original theorem is less adaptable to more exotic expansions like continued fractions:

$$C(n) = \frac{a_1(n)}{1+} \; \frac{a_2(n)}{1+} \; \cdots \; \frac{a_n(n)}{1},$$

which can be expressed as

$$C(n) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(0), \quad \text{with} \quad f_{k,n}(z) = \frac{a_k(n)}{1 + z} .$$

Nevertheless, a unifying principle applicable to a variety of expansions exists if we restrict our attention to scenarios in which all functions $f_n(z)$ and $f_{k,n}$ map simply-connected domain into a compact subset of itself. A simple example of the kinds of possible extensions of classical Tannery's Theorem is the following (theory developed later in this paper):

Example: The continued fraction $\frac{a_1}{1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2}{1} \genfrac{}{}{0pt}{}{}{+ \cdots +} \frac{a_n}{1} \rightarrow \lambda$ as $n \to \infty$, if $|a_k| < \frac{1}{4}$

Suppose $\lim_{n\to\infty} a_k(n) = a_k$ for each $k \le n$ and $|a_k(n)| < \frac{1}{4}$ for all such terms.

Then $$\frac{a_1(n)}{1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2(n)}{1} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{a_n(n)}{1} \rightarrow \lambda \text{ as } n \to \infty$$

A continuous analog of the Tannery Theorem is the following:

Tannery's Theorem (continuous)[2]: [Let $\{f_n(x)\}$ be a sequence of functions continuous on R]. Suppose $\lim\limits_{n\to\infty} f_n(x) = g(x)$ uniformly in any fixed interval, and there exists a positive function M(x) where $|f_n(x)| \le M(x)$ and $\int_a^\infty M(x)dx$ converges.

Then
$$\lim_{n\to\infty}\int_a^n f_n(x)dx = \int_a^\infty g(x)dx \quad .$$

The classical Tannery's Theorem provides the following result:

$$\lim_{n\to\infty}\left(a_1(n)+a_2(n)+\cdots+a_n(n)\right) = \lim_{n\to\infty} a_1(n)+\lim_{n\to\infty} a_2(n)+\cdots+\lim_{n\to\infty} a_n(n)$$

The generalizations described here take the forms:

$$\lim_{n\to\infty} t_{1,n}\circ t_{2,n}\circ\cdots\circ t_{n.n}(z) = \lim_{n\to\infty} t_{1,n}\circ\lim_{n\to\infty} t_{2,n}\circ\cdots\circ\lim_{n\to\infty} t_{n,n}(z)$$

or

$$\lim_{n\to\infty} t_{n,n}\circ t_{n\text{-}1,n}\circ\cdots\circ t_{1,n}(z) = \lim_{n\to\infty} t_{n,n}\circ\lim_{n\to\infty} t_{n-1,n}\circ\cdots\circ\lim_{n\to\infty} t_{1,n}(z)$$

Additional theory addresses scenarios in which the distribution of limits shown above does *not* occur (e.g., the Riemann Integral) . . . making mathematical life a bit more interesting.

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2. **Extending Tannery's Theorem to Inner Composition *with Contractions***

By *contractions* is meant the following *domain contractions*:

Theorem (Henrici [1], 1974). Let f be analytic in a simply-connected region S and continuous on the closure S' of S. Suppose f(S') is a bounded set contained in S. Then $f^n(z) = f\circ f\circ\cdots\circ f(z)\to\alpha$, the *attractive fixed point* of f in S, for all z in S'.

This result can be extended to *forward iteration* (or *inner composition*) involving a sequence of functions:

Theorem 2.1:(*Lorentzen*, [5],1990) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each n, $f_n(D) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ converges uniformly in D to a constant function F(z) = λ.

(Note: This result is sometimes called the *Lorentzen-Gill Theorem* since the second author obtained the result in a specific case in previous papers [4], [6])

The concept underlying *Tannery's Theorem* extends easily to this setting:

Theorem 2.2: (*Gill*, [8],1992) Let $\{f_{k,n}\}$, $1 \le k \le n$ be a family of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each k and n, $f_{k,n}(D) \subset \Omega$ and, in addition, $\lim_{n\to\infty} f_{k,n}(z) = f_k(z)$ uniformly on D for each k. Then , with $F_{p,n}(z) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{p,n}(z)$,

$$F_{n,n}(z) \to \lambda \text{ , a constant function , as } n \to \infty \text{, uniformly on D.}$$

Comment: The condition $\lim_{n\to\infty} f_{k,n}(z) = f_k(z)$, if discarded, allows the possibility of divergence by oscillation: viz.,

$$f_{1,n}(z) = \begin{cases} .5 \text{ if n is odd} \\ -.5 \text{ if n is even} \end{cases} \text{, otherwise } f_{k,n}(z) \equiv \frac{z}{2}, \text{ on } S = (|z| < 1) \text{ .}$$

Proof: Theorem 2.1 defines λ. Write $Z_{p,n} = f_{p+1,n} \circ f_{p+2,n} \circ \cdots \circ f_{n,n}(z)$. Then

$$\begin{aligned} |F_{n,n}(z) - \lambda| &= |F_{p,n}(Z_{p,n}) - \lambda| \\ &\le |F_{p,n}(Z_{p,n}) - F_p(Z_{p,n})| + |F_p(Z_{p,n}) - \lambda| \end{aligned}$$

For the second term in the inequality, choose and fix p sufficiently large that $|F_p(z) - \lambda| < \frac{\varepsilon}{2}$ for all z in D. For the first term, choose n sufficiently large to insure $|F_{p,n}(z) - F_p(z)| < \frac{\varepsilon}{2}$ for all z in D. This is true since a finite composition of functions of the type described above, converging uniformly on D, will also converge uniformly on D. ||

A *Tannery Transformation*: An existing compositional structure $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ may be transformed using these ideas:

Corollary: Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each n, $f_n(D) \subset \Omega$. Now suppose there exists a sequence of functions analytic on D and depending upon both k and n, $\{t_{k,n}\}$, such that $t_{k,n}(D) \subset \Omega$ and $\lim_{n\to\infty} t_{k,n}(z) = z$ uniformly on D, for each k. Then

$$T_n(z) = f_1 \circ t_{1,n} \circ f_2 \circ t_{2,n} \circ \cdots \circ f_n \circ t_{n,n}(z) \;\to\; \lambda ,$$

.

where $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z) \to \lambda$.

Proof: Set $g_{k,n}(z) = f_k(t_{k,n}(z))$ and apply the theorem. ||

Example: *fixed-point continued fractions*

$$C_n(\omega) = \frac{\alpha_1(\alpha_1+1)}{1 \quad +} \;\; \frac{\alpha_2(\alpha_2+1)}{1 \quad +} \;\cdots\; \frac{\alpha_n(\alpha_n+1)}{1 \;+\; \omega}$$

converges under the following stipulations: $|\alpha_n| < \frac{1}{5}$, $|\omega| < \frac{1}{2}$. The $\{\alpha_n\}$ are the *attractive fixed points* of the linear fractional transformations $t_k(\omega) = \frac{\alpha_k(\alpha_k+1)}{1+\omega}$.
Thus, one may write $C_n(\omega) = t_1 \circ t_2 \circ \cdots \circ t_n(\omega)$. (If $\alpha_n \equiv \alpha$, then $\lim_{n\to\infty} C_n(\omega) = \alpha$).

Writing $C_{n,n}(\omega) = \frac{\alpha_1(n)(\alpha_1(n)+1)}{1 \quad +} \;\; \frac{\alpha_2(n)(\alpha_2(n)+1)}{1 \quad +} \;\cdots\; \frac{\alpha_n(n)(\alpha_n(n)+1)}{1 \;+\; \omega}$, where

$\lim_{n\to\infty} \alpha_k(n) = \alpha_k$ for each k, we have $\lim_{n\to\infty} C_{n,n}(\omega) = \lim_{n\to\infty} C_n(\omega)$.

Example: *Nested logarithms* $\frac{1}{2}\mathrm{Ln}\left(2+\frac{1}{3}\mathrm{Ln}\left(3+\frac{1}{4}\mathrm{Ln}\left(4+\cdots\right)\right)\cdots\right)$

Here, $t_k(z) = \frac{1}{k+1}\mathrm{Ln}(k+1+z)$, $|z|<1 \;\Rightarrow\; |t_k(z)| \le \rho < 1$. Thus,

$t_1 \circ \cdots \circ t_n(z) \;\to\; .438699\cdots$. Similarly, $\frac{1}{2}\mathrm{Ln}\left(2\cdot a_1(n)+\frac{1}{3}\mathrm{Ln}\left(3\cdot a_2(n)+\cdots\right)\cdots\right)$, where $1 \le a_k(n) \to 1$, converges to the same limit. The obvious choice for an initial value of z is 0.

Example: Iteration of *Functions defined by Infinite Integrals*

$$t_k(z)=\int_0^\infty \varphi_k(t,z)dt \quad \text{or} \quad t_{k,n}(z)=\int_0^n \varphi_k(t,z)dt$$

For instance: $t_{k,n}(z)=\int_0^n e^{-t(k+1+\varepsilon+z)}dt$, $|z|<1$. Giving rise to a complicated expansion that converges to the limit of the continued fraction:

$$t_{1,n} \circ \cdots \circ t_{n,n}(z) \;\to\; \cfrac{1}{2+\varepsilon+\cfrac{1}{3+\varepsilon+\cfrac{1}{4+\varepsilon\ddots}}}$$

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Extending Tannery's Theorem to Inner Compositions *without Contractions*

Theorem 2.3: (*Gill*, 2011) Consider sequences of polynomials converging to entire functions: $f_{k,n}(z)=z+a_{2,k}z^2+a_{3,k}z^3+\cdots+a_{n,k}z^n \quad \to f_k(z)$ as $n\to\infty$ for k=1,2,3, . . .
Set $\phi_{k,n}(z)=z+a_{2,k}(n)z^2+a_{3,k}(n)z^3+\cdots+a_{n,k}(n)z^n$ where $\lim_{n\to\infty} a_{j,k}(n)=a_{j,k} \quad \forall k,j$, $|a_{j,k}(n)|<\rho_k^{j-1}$ for all n , and $\sum_{k=1}^{\infty}\rho_k<\infty$. Next, set $F_k(z)=f_1\circ f_2\circ\cdots\circ f_k(z)$, where $F(z)=\lim_{k\to\infty}F_k(z)$, and $\Phi_{p,n}(z)=\phi_{1,n}\circ\phi_{2,n}\circ\cdots\circ\phi_{p,n}(z)$, $Z_{p,n}=\phi_{p+1,n}\circ\phi_{p+2,n}\circ\cdots\circ\phi_{n,n}(z)$.
Then

$$\lim_{n\to\infty}\Phi_{n,n}(z)=F(z).$$

Outline of Proof: Consider $|z|\le R$. Then, $|\phi_{k,n}(z)| \le \frac{|z|}{1-\rho_k|z|} \le \frac{R}{1-\rho_k R}$, which may be repeated to give $|Z_{p,n}| \le R_0 = 2R$. The original Tannery's Theorem shows that

$\phi_{k,n}(z) \to f_k(z)$ uniformly on $(|z| \le R_0) = S$. As in Kojima's Theorem [10],

$\left|\phi_{p,n}(z) - z\right| \le \dfrac{\rho_p |z|^2}{1-\rho_p |z|}$, which may be used to prove that

$$\left|Z_{p,n} - z\right| \le 2R_0^2 \cdot \sum_{k=p+1}^{\infty} \rho_k \to 0 \text{ as } p \to \infty .$$

Writing
$$\begin{aligned}\left|\Phi_{n,n}(z) - F(z)\right| \le & \left|\Phi_{p,n}(Z_{p,n}) - F_p(Z_{p,n})\right| \\ & + \left|F_p(Z_{p,n}) - F(Z_{p,n})\right| \\ & + \left|F(Z_{p,n}) - F(Z)\right|\end{aligned}$$

choose p so large that each of the last two expressions are less than $\dfrac{\varepsilon}{3}$ (functions converge uniformly on S) . Then choose n so large that the first is less than $\dfrac{\varepsilon}{3}$.

Therefore $\left|\Phi_{n,n}(z) - F(z)\right| < \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} = \varepsilon$ ||

Theorem 2.4 : Let S be a simply-connected domain and $\{t_{k,n}\}$, $k \le n$, a sequence of functions analytic in S where $t_{k,n}(S) \subset S$ and $t_{k,n}(z) \to t_k(z)$ for each k Suppose that

(a) $\left|t_{k,n}(z) - t_k(z)\right| < \varepsilon_k(n) \to 0$, as $n \to \infty$ for all z in S, and

(b) $\left|t_k(z_1) - t_k(z_2)\right| < \rho_k \left|z_1 - z_2\right|$, $\forall z_1, z_2$ in S .

Then $$\left|t_{1,n} \circ t_{2,n} \circ \cdots \circ t_{n,n}(z) \;-\; t_1 \circ t_2 \circ \cdots \circ t_n(z)\right| \;<\; \sum_{k=1}^{n} \left(\prod_{j=0}^{k-1} \rho_j \right) \varepsilon_k(n)$$

Proof: Set $\Phi_{p,n} = t_{p,n} \circ t_{p+1,n} \circ \cdots \circ t_{n,n}(z)$ and $\Psi_{p,n} = t_p \circ t_{p+1} \circ \cdots \circ t_n(z)$

Then

$$\left|\Phi_{1,n}-\Psi_{1,n}\right| \;\le\; \left|t_{1,n}(\Phi_{2,n})-t_1(\Phi_{2,n})\right| \;+\; \left|t_1(\Phi_{2,n})-t_1(\Psi_{2,n})\right|$$

$$< \;\; \varepsilon_1(n)+\;\rho_1\left|\Phi_{2,n}-\Psi_{2,n}\right|$$

$$< \;\; \varepsilon_1(n)+\rho_1\left[\left|t_{2,n}(\Phi_{3,n})-t_2(\Phi_{3,n})\right|+\left|t_2(\Phi_{3,n})-t_2(\Psi_{3,n})\right|\right]$$

$$\vdots$$

$$< \;\; \sum_{k=1}^{n}\left(\prod_{j=0}^{k-1}\rho_j\right)\varepsilon_k(n)$$

Since the values of $\{\,\rho_k\,\}$ are not necessarily less than one, $\{\,\Psi_{1,n}(z)\,\}$ might diverge and $\{\,\Phi_{1,n}(z)\,\}$ simply track that sequence, assuming $t_{k,n}\to t_k$ rapidly enough. ||

Tannery Continued Fractions

Example: Consider the *Tannery C-Fraction* expansion:

$$\frac{C_1(n)z}{1}\,+\,\frac{C_2(n)z}{1}\,+\cdots+\,\frac{C_n(n)z}{1}$$

Here $t_{k,n}(w)=\dfrac{C_k(n)z}{1+w}$, with $|w|<r<1$, $|z|<R$, and $|C_k(n)|<C$.

$|t_{k,n}(w)|<\dfrac{CR}{1-r}$ with $R<\dfrac{r(1\text{-}r)}{C}$ insures $|t_{k,n}(w)|<r$ when $|w|<r$.

From (a) of Theorem 2.4 , $|t_{k,n}(w)-t_k(w)|<\dfrac{R}{1-r}\;|C_k(n)-C_k|\;<\;\dfrac{R}{1-r}\;\sigma_k(n)$, and

(b) $|t_{k,n}(w_1)-t_{k,n}(w_2)|<\;\dfrac{CR}{(1\text{-}r)^2}\;=\;\rho$. Therefore

$$\left|t_{1,n}\circ t_{2,n}\circ\cdots\circ t_{n,n}(z)\;-\;t_1\circ t_2\circ\cdots\circ t_n(z)\right|\;<\;\frac{R}{1\text{-}r}\;\sum_{k=1}^{n}\rho^{k-1}\sigma_k(n)\,.$$

To illustrate, let $r=\dfrac{1}{2}$, $R=\dfrac{1}{4}$, $C=1$, $\sigma_k(n)=\dfrac{k}{n^3}$. Hence $\rho_k\equiv 1$.

This gives, after a few calculations,

$$\left|\frac{C_1(n)z}{1}\,{}_{+}\,\frac{C_2(n)z}{1}\,{}_{+\cdots+}\,\frac{C_n(n)z}{1} \;-\; \frac{C_1 z}{1}\,{}_{+}\,\frac{C_2 z}{1}\,{}_{+\cdots+}\,\frac{C_n z}{1}\right| < \frac{1}{4n}\left(1+\frac{1}{n}\right) \;\to\; 0\text{, so that}$$

$$F_n(z) = \frac{C_1(n)z}{1}\,{}_{+}\,\frac{C_2(n)z}{1}\,{}_{+\cdots+}\,\frac{C_n(n)z}{1} \;\to\; F(z)\text{, analytic in } |z| < R\,.$$

Example: *Variation on Continuous Analogue of Tannery's Theorem*

Employing Theorem 2.4, set $t_{k,n}(z) = \int_{k-1}^{k} f(x,n)dx + z$, and $t_k(z) = \int_{k-1}^{k} g(x)dx + z$,

with $\lim_{n\to\infty} F(x,n) = g(x)$. Then $\left|t_{k,n}(z) - t_k(z)\right| \le \operatorname*{Max}_{k-1\le x\le k}\left|f(x,n) - g(x)\right| = \varepsilon_k(n)$

and $\left|t_k(z_1) - t_k(z_2)\right| = \left|z_1 - z_2\right| \;\Rightarrow\; \rho_k \equiv 1$. Thus,

$$\left|\int_0^n f(x,n)dx - \int_o^n g(x)dx\right| \le \sum_{k=1}^{n} \varepsilon_k(n)$$

If the sum tends to zero, the first integral grows closer to the second, even if the second integral diverges.

Example: $\int_0^n \sin(x + \sigma(n))dx$, where , e.g., $\sigma(n) < \frac{1}{n^3}$, approximates to any required degree of accuracy the divergent integral $\int_0^n \sin(x)dx$

Repeated Roots: extending Tannery's idea

Corollary: Consider the following composition of roots:

$$R_{n,n} = \sqrt{\sigma_1(n) + \sqrt{\sigma_2(n) + \sqrt{\sigma_3(n) + \cdots + \sqrt{\sigma_n(n)}}}}\ \text{, where}$$

all entries are positive real numbers. When does

$$R_{n,n} \to \lim_{n\to\infty} \sqrt{\sigma_1 + \sqrt{\sigma_2 + \cdots + \sqrt{\sigma_n}}} \quad ?$$

Applying Theorem 2.4 , set $t_{k,n}(z) = \sqrt{\sigma_k(n) + z}$, $z \ge 0$,

$$\sigma_k(n),\ \sigma_k \ \ge\ M(k) > 0, \text{ and } \left|\sigma_k(n) - \sigma_k\right| < \delta_k(n) \;\to\; 0\,.$$

Then conditions (a) and (b) assume the forms

(a) $\left|t_{k,n}(z)-t_k(z)\right| \;\le\; \dfrac{\left|\sigma_k(n)-\sigma_k\right|}{\sqrt{\sigma_k(n)}+\sqrt{\sigma_k}} \;\le\; \dfrac{1}{2M(k)}\left|\sigma_k(n)-\sigma_k\right| < \dfrac{\delta_k(n)}{2M(k)} = \varepsilon_k(n)$,

(b) $\left|t_k(z_1)-t_k(z_2)\right| \le \dfrac{1}{2M(k)}\left|z_1-z_2\right| = \rho_k\left|z_1-z_2\right|$

Hence

$$\left|t_{1,n}\circ\cdots\circ t_{n,n}(0)-t_1\circ\cdots\circ t_n(0)\right| \;<\; \sum_{k=1}^{n}\prod_{j=1}^{k}\frac{1}{M(j)}\cdot\frac{\delta_k(n)}{2^k}\,.$$

Example: $\sqrt{1+\frac{1}{n}+\sqrt{2+\frac{1}{n}+\sqrt{3+\frac{1}{n}+\cdots}}} \;-\; \sqrt{1+\sqrt{2+\sqrt{3+\cdots}}} \;\to\; 0\,.$

We find that

$$\left|t_{1,n}\circ\cdots\circ t_{n,n}(0)-t_1\circ\cdots\circ t_n(0)\right| \;<\; \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k\sqrt{k!}} \;\to\; 0\,.$$

Example: (trivial), from Example 4 above: *Definite Integrals* :

Given $\phi\in C[0,1]$ and $\left|a_k(n) - \frac{1}{n}\,\phi(\frac{k}{n})\right| < \varepsilon_k(n)$, where $\sum_{k=1}^{n}\varepsilon_k(n) \;\to\; 0$. Then

$$\sum_{k=1}^{n}a_k(n) \;\to\; \int_0^1\phi(t)\,dt\;.$$

Example: $\phi(t)=t^2 \;\Rightarrow\; \frac{1}{n}\phi(\frac{k}{n}) \;=\; \frac{k^2}{n^3}$, hence $\sum_{k=1}^{n}\frac{k^2+n}{n^3} \;\to\; \int_0^1 t^2dt = \frac{1}{3}$

Example: *Exponential Expansion*

$$T_{n,n}(z)=t_{1,n}\circ t_{2,n}\circ\cdots\circ t_{n,n}(z)\ ,\ \text{where}\ \ t_{k,n}(z)=\frac{z}{n}+\frac{1}{6}e^{kz/n}\ ,\ n\ge 2\ ,\ |z|\le 1\,.$$

Hence $\left|t_{k,n}(z)\right|<1$, $\rho_k\equiv 0$, and $\varepsilon_1(n)<\dfrac{5}{3n}\to 0$.

Thus $t_{k,n}(z) \;\to\; t_k(z)\equiv\dfrac{1}{6}$ and Theorem 7 $\Rightarrow$ $T_{n,n}(z)\to\dfrac{1}{6}$.

. . .

Related to Theorem 2.4 is the following result:

Theorem 2.5 (*Gill* 2011) Consider the nested sets $S = (|z| \le R)$, $S_1 = (|z| \le R_1)$, $S_2 = (|z| \le R_2)$ where $R_1 = R + \frac{C\rho}{1-\rho}$ and $R_2 = R + \frac{2C\rho}{1-\rho}$. Let $\{f_{k,n}\}$ be a family of functions analytic on S_2 and $0 \le \rho < 1$, where:

(1) $|f_{k,n}(z) - z| \le C\rho^k$ on S_2 <u>and</u> (2) $f_{k,n}(z) \to f_k(z)$ as $n \to \infty$, uniformly on S_2.

Set $Z_{p,n} = f_{p+1,n} \circ f_{p+2,n} \circ \cdots \circ f_{n,n}(z)$, $F_{p,n}(z) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{p,n}(z)$, and $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$. Then there exists a function F(z) analytic on S_1 such that

$$F_n(z) \to F(z) \text{ uniformly on } S_1 \ \underline{\text{and}} \ F_{n,n}(z) \to F(z) \text{ uniformly on } S.$$

Sketch of proof: Theorem 2.6 [11] shows that $F_n(z) \to F(z)$ uniformly on S_1. Next, for $z \in S$,

$$\begin{aligned} &|f_{n,n}(z)| \le |z| + C\rho^n \le R + C\rho^n \\ &|f_{n-1,n} \circ f_{n,n}(z)| \le R + C\rho^n + C\rho^{n-1} \\ &\quad \vdots \\ &|Z_{p,n}| \le R_1 \qquad\qquad \text{I.e., } Z_{p,n} \in S_1 . \end{aligned}$$

Now write $|F_{n,n}(z) - F(z)| \le |F_{p,n}(Z_{p,n}) - F_p(Z_{p,n})| + |F_p(Z_{p,n}) - F(z)|$

Fix p so large that the second term on the right is $< \frac{\varepsilon}{2}$. If n is sufficiently large the first term on the right side is $< \frac{\varepsilon}{2}$. Hence, for $z \in S$, $F_{n,n}(z) \to F(z)$ on S. ||

3. **Extending Tannery's Theorem to Outer Composition *with Contractions***

Theorem 3.1**:** (Henrici [1], 1974). Let f be analytic in a simply-connected region S and continuous on the closure S' of S. Suppose f(S') is a bounded set contained in S. Then $f^n(z) = f \circ f \circ \cdots \circ f(z) \to \alpha$, the attractive fixed point of f in S, for all z in S'.

This fundamental result for *contraction mappings* can be extended to an infinite composition of functions arranged as *backward iteration* (or *outer composition*) :

Theorem 3.2 : [*Gill*, [7],1991) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D. Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ for all n. Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Then $G_n(z) \to \alpha$ uniformly on the closure of D *if and only if* the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to the number α.

comment: The existence of the $\{\alpha_n\}$ is guaranteed by Theorem 1. Note the simple counter-example $g_n(z) = -.5$ for n odd and $g_n(z) = .5$ for n even, in the unit disk (|z|<1). It is not essential that $g_n \to g$, although that is usually the case. If $g_n \to g$, then $\alpha_n \to \alpha$.

Example: let $G(z) = \cfrac{\frac{e^z}{4}}{3+z} + \cfrac{\frac{e^z}{8}}{3+z} + \cfrac{\frac{e^z}{12}}{3+z} + \cdots$, where $|z| \le 1$. We solve the *continued fraction equation* $G(\alpha) = \alpha$ in the following way: Set $t_n(\xi) = \dfrac{e^z/4n}{3+z+\xi}$; let $g_n(z) = t_1 \circ t_2 \circ \cdots \circ t_n(0)$. Now calculate $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$, starting with $z = 1$. One obtains $\alpha = .087118118\cdots$ to ten decimal places after ten iterations [7].

Next, consider a sequence of functions $\{g_{k,n}\}$ dependent upon both k and n and defined on a suitable domain D. Define $G_{p,n}(z) = g_{p,n} \circ g_{p-1,n} \circ \cdots \circ g_{1,n}(z)$, with $p \le n$.

Theorem 3.3: Suppose $\{g_{k,n}\}$, with $k \le n$, is a family of functions analytic on a simply-connected domain D and continuous on its closure, with $g_{k,n}(D) \subset \Omega$, a compact subset of D, for all k and n. Then

$G_{n,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z) \to \alpha$ uniformly on the closure of D

if and only if

the sequence of fixed points $\{\alpha_{k,n}\}$ of $\{g_{k,n}\}$ converge* to α.

Comments: When $\lim_{n\to\infty} g_{k,n}(z) = g_k(z)$, for each value of k, both sequences converge to the limit described in theorem 2. * For ε>0 $\exists$ N=N(ε) $\ni$ N<k ≤ n $\Rightarrow$ $|\alpha_{k,n} - \alpha| < \varepsilon$

Proof: The proof (of sufficiency) is similar to that of theorem 3.2.

Set $D = (|z|<1)$. Let $\Phi(z) := \dfrac{z-\alpha}{1-\bar{\alpha}z}$. Then $\Phi : D \to D$ is analytic there, with $\Phi(\alpha) = 0$ and $\Phi^{-1}(0) = \alpha$. Set $q_{k,n}(z) = \Phi \circ g_{k,n} \circ \Phi^{-1}(z)$.

Lemma : $q_{k,n}(0) \to 0$ as both $k, n \to \infty$.

Proof of Lemma: This result will follow if $g_{k,n}(\alpha) \to \alpha$. Write

(1) $|g_{k,n}(\alpha) - \alpha| \le |g_{k,n}(\alpha) - g_{k,n}(\alpha_{k,n})| + |\alpha_{k,n} - \alpha|$

For $\varepsilon > 0$, choose K and N such that k>K and n>N imply each term of the right side of (1) is less than $\frac{\varepsilon}{2}$. This is possible for the first term because the $\{g_{k,n}\}$ are uniformly bounded on D, thus equicontinuous there. Hence $q_{k,n}(0) \to 0$ as $k, n \to \infty$.

Now, the existence of the compact set Ω implies $|g_{k,n}(z)| \le \mu < 1$ for all z in D. Thus

(2) $$\operatorname*{Sup}_{k,n}(\operatorname*{Sup}_{|z|<1} |q_{k,n}(z)| = \rho < 1 \qquad \text{exists.}$$

Since $q_{k,n}(0) \to 0$ as $k, n \to \infty$, there exists a sequence $\{\varepsilon_{k,n}\}$ such that $0 \le \varepsilon_{k,n} \to 0$ as $k, n \to \infty$, and $|q_{k,n}(0)| \le \varepsilon_{K,N}$ for all k>K and n>N. (E.g., set $\varepsilon_{K,N} = \operatorname*{Sup}_{k>K, n>N} |q_{k,n}(0)|$)

Set $H_{k,n}(z) = \dfrac{q_{k,n}(z)}{\rho}$. Then $|H_{k,n}(z)| < 1$ for all |z|<1. An application of Schwartz's Lemma [3] gives

$$|H_{k,n}(z)| \le \frac{|H_{k,n}(0)| + |z|}{1 + |H_{k,n}(0)| \cdot |z|} \le |H_{k,n}(0)| + |z| .$$

Therefore

(3) $$|q_{k,n}(z)| \le |q_{k,n}(0)| + |z|\rho \ .$$

Next, set $Q_{k,n}(z) = q_{k,n} \circ q_{k-1,n} \circ \cdots \circ q_{1,n}(z)$ for all k and n. Then from (2),

(4) $$|Q_{k,n}(z)| < \rho < 1 \ \text{ for all k and n.}$$

Writing p = n + m , begin an inductive procedure with an arbitrary but large value of n, with the goal of proving that $\left|Q_{p,p}(z)\right| \to 0$ as p tends to infinity. Employing backward recursion, using (3) and (4):

$$
\begin{aligned}
\left|Q_{n+m,n+m}(z)\right| = \left|q_{n+m,n+m}(Q_{n+m-1,n+m}(z))\right| &\le \left|q_{n+m,n+m}(0)\right| + \rho\left|Q_{n+m-1,n+m}(z)\right| < \varepsilon_{n,n} + \rho\left|Q_{n+m-1,n+m}(z)\right| \\
&\le \varepsilon_{n,n} + \rho\{\left|q_{n+m-1,n+m}(0)\right| + \rho\left|Q_{n+m-2,n+m}(z)\right|\} \\
&< \varepsilon_{n,n} + \rho\varepsilon_{n,n} + \rho^2\left|Q_{n+m-2,n+m}(z)\right| \\
&\le \varepsilon_{n,n}(1+\rho) + \rho^2\{\left|q_{n+m-2,n+m}(0)\right| + \rho\left|Q_{n+m-3,n+m}(z)\right|\} \\
&< \varepsilon_{n,n}(1+\rho) + \varepsilon_{n,n}\rho^2 + \rho^3\left|Q_{n+m-3,n+m}(z)\right| \\
&\le \varepsilon_{n,n}(1+\rho+\rho^2) + \rho^3\{\left|q_{n+m-3,n+m}(0)\right| + \rho\left|Q_{n+m-4,n+m}(z)\right|\} \\
&< \varepsilon_{n,n}(1+\rho+\rho^2+\rho^3) + \rho^4\left|Q_{n+m-4,n+m}(z)\right| \\
&\;\;\vdots \\
&< \frac{\varepsilon_{n,n}}{1-\rho} + \rho^{m-1}\left|Q_{n+1,n+m}(z)\right| \quad < \quad \frac{\varepsilon_{n,n}}{1-\rho} + \rho^m
\end{aligned}
$$

Thus, if n and m are large enough (p is large enough) both terms of the last expression can be made as small as one wishes. Hence $\left|Q_{p,p}(z)\right| \to 0$ as p tends to infinity. It follows immediately that $G_{n,n}(z) \to \alpha$ for all z in D. It is a simple matter to extend these results to more general simply-connected domains, D, by using appropriate Riemann Mapping Functions. ǁ

Example: The modified *fixed-point continued fraction* seen before

$$C_n(\omega) = \frac{\alpha_1(\alpha_1+1)}{1 \;\; +} \;\; \frac{\alpha_2(\alpha_2+1)}{1 \;\; +} \;\cdots\; \frac{\alpha_n(\alpha_n+1)}{1 \; + \; \omega}$$

can be reconfigured to give a *modified reverse fixed-point continued fraction*:

$$G_n(\omega) = \frac{\alpha_n(\alpha_n+1)}{1 \;\; +} \;\; \frac{\alpha_{n-1}(\alpha_{n-1}+1)}{1 \;\; +} \;\cdots\; \frac{\alpha_1(\alpha_1+1)}{1 \; + \; \omega}$$

convergent when $\left|\alpha_n\right| < \frac{1}{5}$, $\left|\omega\right| < \frac{1}{2}$, and $\lim_{n\to\infty} \alpha_n = \alpha$. The $\{\alpha_n\}$ are the *attractive fixed points* of the linear fractional transformations $t_k(\omega) = \dfrac{\alpha_k(\alpha_k+1)}{1+\omega}$.

Thus, one may write $G_n(\omega) = t_n \circ t_{n-1} \circ \cdots \circ t_1(\omega) \;\; \to \;\; \alpha$, as $n \to \infty$.

Setting $G_{n,n}(\omega) = \cfrac{\alpha_n(n)(\alpha_n(n)+1)}{1 \quad +} \quad \cfrac{\alpha_{n-1}(n)(\alpha_{n-1}(n)+1)}{1 \quad +} \ \cdots \ \cfrac{\alpha_1(n)(\alpha_1(n)+1)}{1 \ + \ \omega}$,
where $\lim_{k,n\to\infty} \alpha_k(n) = \alpha$, we have

$$\lim_{n\to\infty} G_{n,n}(\omega) = \lim_{n\to\infty} G_n(\omega) = \alpha .$$

The following result extends the scope of Theorem 3.3 somewhat:

Theorem 3.3a: Suppose $\{g_{k,n}(\zeta,z)\}$, with $k \le n$, is a sequence of functions analytic with respect to z on a simply-connected domain, D, for each $\zeta \in S$, a second simply-connected domain, with $g_{k,n}(S,D) \subset \Omega$, a compact subset of D, for all k and n. Let the sequence of fixed points $\{\alpha_{k,n}(\zeta)\}$ of $\{g_{k,n}\}$ converge to $\alpha(\zeta)$ uniformly on S. (I.e., $\alpha_{k,n} \to \alpha$ as both k and $n \to \infty$, with $k \le n$). Then

$$G_{n,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z) \to \alpha(\zeta) \quad \text{uniformly on } S \times D$$

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4. **Extending Results for Outer Composition *without Contractions***

Theorem 4.1: Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D. Let $g_n(D) \subset D$ for all n. Suppose there exists a sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in D converging to a number α, and
(1) $|g_k(z) - \alpha_k| \le \rho_k |z - \alpha_k|$, $0 \le \rho_k < 1$ and (2) $|\alpha\text{-}\alpha_k| < \varepsilon_k \to 0$

then , setting $H_{n,n+p}(z) = g_{n+p} \circ g_{n+p-1} \circ \cdots \circ g_n(z)$,

$$\left|H_{n,n+p}(z) - \alpha\right| \le |z-\alpha| \cdot \prod_{k=n}^{n+p} \rho_k + 2\sum_{k=n}^{n+p-1} \varepsilon_k \cdot \prod_{j=k+1}^{n+p} \rho_j + 2\varepsilon_{n+p}$$

In particular $|G_n(z) - \alpha| = |H_{1,n}(z) - \alpha| = |z-\alpha| \prod_1^n \rho_k + 2\sum_1^{n-1}\left(\varepsilon_k \prod_{k+1}^n \rho_j\right) + 2\varepsilon_n$

Proof: The repeated application of the inequality

$|g_n(z)-\alpha| \le |g_n(z)-\alpha_n| + |\alpha-\alpha_n|$ and use of the fact that $\rho_k < 1$ are sufficient. ||

Comment: The simple example in which $\rho_k = 1-\frac{1}{k}$ and $\varepsilon_k = \frac{1}{k^2}$ for D=(|z|<1) lies outside the context of Theorem 3.2 , and yields, after simplification,

$$|G_n(z)-\alpha| < \frac{1}{n+1}|z-\alpha| + \frac{2}{n+1}\left\{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\right\} + \frac{2}{n^2} \;\to\; 0 \text{ as } n\to\infty$$

Example: Let $g_k(z) = \frac{\alpha_k(\alpha_k+1)}{1+z}$, with $D = \{z{:}|z|<\frac{1}{2}\}$.

Set

$\alpha_k = \frac{\sqrt{2}-1}{2} - \frac{1}{k+4}$. Then $|\alpha\text{-}\alpha_k| = \frac{1}{k+4}$ and $g_k(D) \subseteq D$. And

$\rho_k = \left|\frac{\alpha_k}{1+z}\right| < \frac{1}{2}$, $g_k(z) \to g(z) = \frac{1/4}{1+z}$. Thus, after simplification,

$$|G_n(z)-\alpha| < \frac{1}{2^n}|z-\alpha| + 2\left\{\frac{1}{2(n+3)} + \frac{1}{2^2(n+2)} + \cdots \frac{1}{2^{n-1}(5)}\right\} + \frac{2}{n+4} \;=\; E(n)$$

Applying the original Tannery's Theorem to the series component shows E(n) tends to zero as n becomes infinite.

Extending the result above to *Tannery continuous compositions,* we have

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Theorem 4.2 **:** Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D. Let $g_n(D) \subset D$ for all n. Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Suppose there exists a sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in D converging to a number α, and suppose $\{g_{k,n}\}$, with $k \le n$, is a sequence of functions analytic on a simply-connected domain D, with $g_{k,n}(D) \subset D$, and $G_{p,n}(z) = g_{p,n} \circ g_{p-1,n} \circ \cdots \circ g_{1,n}(z)$. Assume further

(1) $\left| g_{k,n}(z) - g_k(z) \right| < \sigma_k(n) \to 0$ as $n \to \infty$

(2) $\left| g_n(\zeta_1) - g_n(\zeta_2) \right| < \rho_n \left| \zeta_1 - \zeta_2 \right|$, $0 \le \rho_n < 1$

(3) $\left| \alpha - \alpha_n \right| < \varepsilon_n \to 0$

Then $\left| G_{n,n}(z) - \alpha \right| \le \left| z - \alpha \right| \cdot \prod_{k=1}^{n} \rho_k + \sum_{k=1}^{n-1} \eta_k(n) \cdot \prod_{j=k+1}^{n} \rho_j + \eta_n(n)$,
where $\eta_k(n) = \sigma_k(n) + 2\varepsilon_k$

Proof: Write $\left| G_{n,n}(z) - \alpha \right| \le \left| G_{n,n}(z) - G_n(z) \right| + \left| G_n(z) - \alpha \right|$

It is easily seen that

(4) $\left| G_{n,n}(z) - G_n(z) \right| < \sigma_n(n) + \sum_{k=1}^{n-1} \left(\prod_{j=k+1}^{n} \rho_j \right) \sigma_k(n)$

And, from the previous theorem,

$$\left| G_n(z) - \alpha \right| \le \left| z - \alpha \right| \cdot \prod_{k=1}^{n} \rho_k + 2 \sum_{k=1}^{n-1} \varepsilon_k \cdot \prod_{j=k+1}^{n} \rho_j + 2\varepsilon_n , \quad \text{so that}$$

$$\left| G_{n,n}(z) - \alpha \right| \le \left| z - \alpha \right| \cdot \prod_{k=1}^{n} \rho_k + \sum_{k=1}^{n-1} \eta_k(n) \cdot \prod_{j=k+1}^{n} \rho_j + \eta_n(n) , \ \eta_k(n) = \sigma_k(n) + 2\varepsilon_k \ \|$$

Example: Let $g_k(z) = \sin(\frac{k\pi}{2} + z)$, $g_{k,n}(z) = \sin(\frac{k\pi}{2} + \frac{k}{n^3} + z)$, $-1 \le z \le 1$

Then $\sigma_k(n) = \frac{k}{n^3}$ and $\rho_k \equiv 1$ $\sigma_k(n) = \frac{k}{n^3}$ and $\rho_k \equiv 1$. From (4) above,

$$\left| \sin(\frac{n\pi}{2} + \frac{n}{n^4} + \sin(\frac{(n-1)\pi}{2} + \frac{n-1}{n^4} + \cdots) - \sin(\frac{n\pi}{2} + \sin(\frac{(n-1)\pi}{2} + \cdots) \right| < \frac{1}{2n}(1 + \frac{1}{n}) \to 0$$

Although neither sequence converges.

Another result:

Theorem 2.7: (Gill, [11] 2011) Let $\{g_n\}$ be a sequence of complex functions defined on $S_0=(|z|\le R_0)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty}\rho_n<\infty$ and $|g_n(z)-z|<C\rho_n$ if $|z|\le R_0$. Set $\sigma=C\sum_1^{\infty}\rho_k$ and $R_0=R+\sigma$. Then, for every $z\in S=(|z|\le R)$, $G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\ \to\ G(z)$, uniformly on compact subsets of S.

Which can be extended to another Tannery result:

Theorem 2.8: (*Gill* 2011) Suppose the functions $\{g_{k,n}\}$ are defined on $S_0=(|z|\le R_0)$, $\{\rho_n\}$ are positive, $\sigma=C\sum_{k=1}^{\infty}\rho_k$ converges, and $S=(|z|\le R)$, $R+\sigma=R_0$. Assume (1) $g_{k,n}(z)\to g_k(z)$ uniformly on S_0 and (2) $|g_{k,n}(z)-z|\le C\rho_k$ there as well.
Then for $z\in S$,

$$\lim_{n\to\infty}G_{n,n}(z)=G(z)=\lim_{n\to\infty}G_n(z).$$

Outline of Proof: Theorem 2.7 shows $G_n=G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\to G(z)$ and (2) may be used repeatedly to show $|g_{p,n}\circ g_{p-1,n}\circ\cdots\circ g_{1,n}(z)|\le R_0$.
Now, write $|G_{n,n}-G|\le|G_{n,n}-G_n|+|G_n-G|$, in which

$$\begin{aligned}|G_{n,n}-G_n|&\le\sum_{k=p+1}^{n}|G_{k,n}-G_{k-1,n}|+\sum_{k=p+1}^{n}|G_k-G_{k-1}|+|G_{p,n}-G_p|\\&\le 2\sum_{k=p}^{\infty}\rho_k+|g_{p,n}(G_{p-1,n})-g_p(G_{p-1,n})|+|g_p(G_{p-1,n})-g_p(G_{p-1})|\\&<2\cdot\frac{\varepsilon}{12}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{2}\ \text{ if p is chosen large enough to insure }\sum_{k=p}^{\infty}\rho_k<\frac{\varepsilon}{12},\end{aligned}$$

and then n is large enough to guarantee each of the last two expressions is $<\frac{\varepsilon}{6}$.
($\{g_{k,n}\}$ and $\{g_k\}$ equicontinuous on S_0, and (1) above)

Thus, for large n, $\left|G_n - G\right| < \dfrac{\varepsilon}{2}$ and the proof is complete. ǁ

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5. **Tannery Theory Potpourri . . . Trivia and Such**

Comment : Consider $T_{n,n}(z) = t_{1,n} \circ t_{2,n} \circ \cdots \circ t_{n,n}(z)$ where each function is of the form $t_{k,n}(z) = a_k(n) + z$ and $\lim\limits_{n\to\infty} a_k(n) = a_k$. Then

$$T_{n,n}(0) = a_1(n) + a_2(n) + \cdots + a_n(n) \quad .$$

Tannery's original theorem covered this sort of thing, using uniform convergence properties:

$$\lim_{n\to\infty}[a_1(n) + a_2(n) + \cdots + a_n(n)] = a_1 + a_2 + \cdots$$

Several examples where TT may or may not apply:

In each instance, $a_k(n) \to a_k \equiv 0$ as $n \to \infty$. Does $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k(n) = 0$?

Example 1: $a_k(n) = \dfrac{k}{n} \Rightarrow \lim\limits_{n\to\infty} T_{n,n}(0) = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k(n) = \infty$

Example 2: $a_k(n) = \dfrac{k}{n^2} \Rightarrow \lim\limits_{n\to\infty} T_{n,n}(0) = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k(n) = \dfrac{1}{2}$

Example 3: $a_k(n) = \dfrac{k}{n^3} \Rightarrow \lim\limits_{n\to\infty} T_{n,n}(0) = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k(n) = a_1 + a_2 + \cdots = 0 + 0 + \cdots = 0$

Example 4: $a_k(n) = \dfrac{1}{n} f\left(\dfrac{k}{n}\right)$, $f \in C[0,1] \Rightarrow \lim\limits_{n\to\infty} T_{n,n}(0) = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k(n) = \int_0^1 f(x)dx$

Simple observations arising from the preceding examples . . .

Theorem 5.1 : Suppose $\lim_{n\to\infty} a_k(n) \equiv 0$ for $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$. Then

(a) $a_k(n) \geq \dfrac{\rho}{n} \;\Rightarrow\; S(n) \geq \rho$

(b) $a_k(n) \leq \dfrac{1}{n^{1+\alpha}}$, $\alpha>0 \;\Rightarrow\; S(n) \to 0$

(c) $a_{k+1}(n) \geq \rho \cdot a_k(n)$, $\rho>1$, $a_1(n) \geq \dfrac{m}{\rho^{n-1}} \;\Rightarrow\; S(n) \geq m>0$

(d) $a_{k+1}(n) \leq \rho \cdot a_k(n)$, $\rho < 1 \;\Rightarrow\; S(n) \to 0$

In more general settings:

Example 5: $a_k(n) = \begin{cases} 0 & \text{if } k<n \\ 1 & \text{if } k=n \end{cases}$ shows that $\lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n)$ may exist in the absence of the condition $a_n(n) \to 0$.

Example 6: $a_k(n) = \begin{cases} 1 & \text{if } k<n \\ 0 & \text{if } k=n \end{cases}$ shows that $a_n(n) \to 0$ does not imply

$\lim_{n\to\infty} T_{n,n}(0) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k(n)$ exists (in a finite sense).

Observe the *Tannery Series*

$$S(n) = a_1(n) + a_2(n) + \cdots + a_n(n), \text{ with}$$

$|a_k(n) - a_k| < \varepsilon_{k,n} \leq \dfrac{\lambda(k)}{n^{\beta}}$, where $\lambda(k)$ is a linear function of k and $\beta > 2$.

Then $\lim_{n\to\infty} S(n) = a_1 + a_2 + \cdots$, provided $\sum a_k$ converges.

Tighter conditions are possible, but this simple example shows that the original Tannery's Theorem for series has more latitude.

Alternating Tannery Series

Alternating series $S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$ require only that

$a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$ for convergence, so it would seem reasonable that a Tannery Series, $S(n) = a_1(n) - a_2(n) + \cdots + (-1)^{n+1} a_n(n)$, in order to converge to the alternating series, should exhibit a fairly rapid convergence of individual terms to those of the series. Theorem 2.4 is applicable ($t_{k,n}(z) = a_k(n) + z$) in that

$|a_k(n) - a_k| < \varepsilon_k(n)$ with $\sum_1^n \varepsilon_k(n) \to 0$ is sufficient to insure the convergence of the

alternating Tannery Series: $|S(n) - S_n| < \sum_1^n \varepsilon_k(n) \to 0$

Example: $S(n) = \dfrac{n^2}{1+n^2} - \dfrac{2n^2}{1+4n^2} + \dfrac{3n^2}{1+9n^2} - \cdots + (-1)^{n+1} \dfrac{n \cdot n^2}{1+n^2 \cdot n^2}$. Here the corresponding *alternating series* is $S_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + (-1)^{n+1} \dfrac{1}{n}$

We find that $|a_k(n) - a_k| < \dfrac{1}{n^2}$ so that $|S(n) - S_n| < \dfrac{1}{n} \to 0$.

An Integral Test for Tannery Series

The following result is an analogue of the familiar Integral Test for series. It doesn't provide a spectacular new perspective on the subject . . . it's merely a curiosity that could probably be improved:

Theorem 5.3 : Let $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$, and suppose that there exists a non-negative, bounded and differentiable function f(x,t) , defined for $x \geq 1$ and $t \geq x$, with $f(k,n) = a_k(n)$, $f_t(x,t) > 0$ and $f_x(x,t) < 0$, $f(1,n) \leq M$. Then

$$\lim_{n\to\infty} S(n) \quad \text{exists if and only if} \quad \lim_{n\to\infty} \int_1^n f(x,n)\,dx \quad \text{exists}$$

Proof: A graphical representation shows the following:

$$\int_1^n f(x,n)\,dx + a_n(n) \leq S(n) \leq \int_1^n f(x,n)\,dx + a_1(n)$$

The hypotheses imply $S(n)$ and $\int_1^n f(x,n)\,dx$ are monotonic increasing. If $S(n)$ converges, $a_n(n)$ tends to zero. . . ||

Two simple examples illustrate the theorem:

Example:

$$S(n)=\frac{n}{(n+1)1^2}+\frac{n}{(n+1)2^2}+\cdots+\frac{n}{(n+1)n^2}=\frac{n}{n+1}(1+\frac{1}{4}+\cdots+\frac{1}{n^2})\to\frac{\pi^2}{6}$$

Here, $\frac{n}{n+1}\int_1^n \frac{1}{x^2}dx+\frac{n}{n+1} \quad \to \quad 1+1=2 \;\geq\; \frac{\pi^2}{6}$

Example: $S(n)=\frac{n}{n+1+1^2}+\frac{n}{n+1+2^2}+\cdots+\frac{n}{n+1+n^2} \quad (\geq \sum_{k=1}^{n}\frac{1}{2+k} \;\to\; \infty)$

Here, $\int_1^n \frac{n}{n+1+x^2}dx=\frac{n}{\sqrt{n+1}}\left(\operatorname{Arc}\tan\frac{n}{\sqrt{n+1}}-\operatorname{Arc}\tan\frac{1}{\sqrt{n+1}}\right) \;\to\; \infty$,

so that $\lim_{n\to\infty}S(n)=\infty$.

Other Simple Results for Tannery Series

Theorem 5.4 : Let $S(n)= a_1(n)+a_2(n)+\cdots+a_n(n)$, and suppose there exists a non-negative, bounded, and differentiable function $f(x,t)$ defined for $x\geq 0$ and $t\geq x$, with $f(k,n)=a_k(n)$. Define $\phi(x)=f(x,x)$. Suppose
$f_x(x,t)\geq 0$, $f_t(x,t)<0$, and $\phi(x)\to 0$ as $x\to\infty$. Also stipulate $f(1,t)\to 0$ as $t\to\infty$.
Then

$$0\leftarrow a_1(n) \;\leq\; S(n) - \int_1^n f(x,n)dx \;\leq\; a_n(n) \;\to\; 0 \quad \text{as n becomes infinite.}$$

Proof: The easiest proof involves drawing a simple histogram. ||

Example: $f(x,t)=\frac{x}{t^2}$, then $f_x(x,t)=\frac{1}{t^2}>0$, $f_t(x,t)=\frac{-2k}{t^3}<0$ and $\phi(x)=\frac{1}{x} \;\to 0$.
The theorem shows that $\frac{1}{2}\left(1-\frac{1}{n^2}\right)+\frac{1}{2}$ - $S(n) \;\to\; 0$, or $\lim_{n\to\infty}S(n)=\frac{1}{2}$, as is easily verified by evaluating $S(n)$ directly: $S(n)=\frac{1}{n^2}\left(1+2+3+\cdots+n\right)$.

Example: A slightly more sophisticated example is the following:

$$S(n) = \frac{1}{1^2+n^2} + \frac{2}{2^2+n^2} + \cdots + \frac{n}{n^2+n^2}$$

Here $f(x,t) = \dfrac{x}{x^2+t^2}$ and the conditions of the theorem are satisfied for the relevant values of the variables. Thus

$$\frac{1}{2}\mathrm{Ln}\left(2-\frac{2}{n}+\frac{1}{n^2}\right)+\frac{1}{2n} - S(n) \to 0, \quad \text{or } S(n) \to \frac{1}{2}\mathrm{Ln}(2)$$

The convergence is very slow. (elementary techniques also show this result)

Example: $S(n) = \dfrac{1}{\sqrt{1+n^4}} + \dfrac{2}{\sqrt{4+n^4}} + \dfrac{3}{\sqrt{9+n^4}} + \cdots + \dfrac{n}{\sqrt{n^2+n^4}}$

Hence $$\int_0^{n-1} \frac{x}{\sqrt{x^2+n^4}}\,dx \to \frac{1}{2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{k^2+n^4}}$$

Theorem 5.5 : Let $S(n) = a_1(n)+a_2(n)+\cdots+a_n(n)$, and suppose there exists a non-negative, bounded, and differentiable function $f(x,t)$ defined for $x \geq 0$ and $t \geq x$, with $f(k,n) = a_k(n)$. Define $\phi(x) = f(x,x)$. Suppose
$f_x(x,t) < 0$, $f_t(x,t) < 0$, and $\phi(x) \to 0$ as $x \to \infty$, and $f(1,t) \to 0$ as $t \to \infty$
Then

$$0 \leftarrow a_n(n) \leq S(n) - \int_1^n f(x,n)dx \leq a_1(n) \to 0 \quad \text{as } n \text{ becomes infinite.}$$

Proof: Draw a picture! ||

Example:

$$S(n) = \frac{n}{1^2+n^2} + \frac{n}{2^2+n^2} + \ldots + \frac{n}{n^2+n^2}$$

The conditions of Theorem 4 are satisfied, giving $\lim_{n\to\infty} S(n) = \dfrac{\pi}{4}$.

(elementary techniques also show this result).

Absolute Convergence & Analytic Functions

Tannery series $a_1(n)+a_2(n)+\cdots+a_n(n)$ are much more interesting if $a_k(n)\to 0$ as $n\to\infty$ for each k <u>and</u> Tannery's Theorem – the original version – does *not* apply. If , in the Tannery Series $S(n)=a_1(n)+a_2(n)+\cdots+a_n(n)$, each $a_k(n)=a_k$, then $S(n)$ is merely a normal series and consequently its *absolute convergence* implies normal convergence. But if this is not the case , and each term involves n, then absolute convergence does not imply convergence, as seen in the following simple example.

Example: Set $a_k(n)=\left\{\begin{matrix} \frac{k}{n^2} \text{ if } n \text{ even} \\ \frac{-k}{n^2} \text{ if } n \text{ odd} \end{matrix}\right\}$. Then, for *n even* $S(n)\to\frac{1}{2}$, but for *n odd* $S(n)\to-\frac{1}{2}$, although the series converges absolutely to the value ½. Here $a_k(n)\to 0$ as $n\to\infty$ for each k . Observe that were the original Tannery Theorem applicable, we would have a very uninteresting $S(n)\to 0$. However, the absolute convergence of $S(n)$ does imply the existence of at least one subsequence $\{n_j\}$ with $\{S_{n_j}\}$ converging, since $|S(n)|\le |a_1(n)|+\cdots+|a_n(n)| \le M$.

Analytic Functions . . .

Theorem 5.6 : Define $F_n(z)=a_{1,n}(z)+a_{2,n}(z)+\cdots+a_{n,n}(z)$ where each term in the Tannery series is analytic on D=(|z|<1) and there exists a positive M such that $|a_{k,n}(z)|\le\frac{M}{n}$ for each $k\le n$. Furthermore, assume there exists an interval (a,b) in (-1,1) so that for each x in this interval, $F_n(x)\to\lambda(x)$. Then $F_n(z)\to\lambda(z)$, analytic on D, uniformly on compact subsets of D.

Proof: Follows immediately from the *Stieltjes-Vitali Theorem* [9], with $\left|F_n(z)\right| < M$. The domain D may of course be generalized. ǁ

Corollary: Set $a_{k,n}(z) = \frac{1}{n}\Phi_{k,n}(z)$ where each $\Phi_{k,n}$ is analytic on D and $\left|\Phi_{k,n}(z)\right| < M$. Then $F_n(z) \to \lambda(z)$, analytic on D, uniformly on compact subsets of D.

Example: Define $F_n(z) = \frac{1}{n}e^{\frac{1+z}{n}} + \cdots + \frac{1}{n}e^{\frac{n+z}{n}}$ with D=(|z|<1). It is easily shown that

$$\left|a_{k,n}(z)\right| < \frac{10}{n} \text{ so that } \left|F_n(z)\right| < 10.$$

Let $z = u + iv$ with 0<u<1, for example.

Writing $f(x,t) = \frac{1}{t}e^{\frac{x+u}{t}}$: $f_x \geq 0$, $f_t < 0$, $f(x,x) \to 0$ and $f(1,t) \to 0$, so that Theorem 8 applies , with $\lim_{n\to\infty} S_n(u) = \lim_{n\to\infty}\int_1^n \frac{1}{n}\, e^{\frac{x+u}{n}}\,dx = e-1$. Therefore, $F_n(z) \to \lambda(z) = e - 1$.

Inner & Outer Composition in a compact set – Convergence to analytic functions

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Theorem 5.7 : Let D=(|z|<1) and S be a simply-connected domain. Suppose there exists a sequence of functions $\{f_{k,n}(\zeta,z)\}_{k\leq n}$ analytic on both D and S, with $f_{k,n}(S,D) \subset \Omega$, compact, $\subset D$. Suppose $f_{k,n} \to f_k$ uniformly on $S\times D$ as $n \to \infty$.

Set (1) $F_1(\zeta,z) = f_1(\zeta,z)$, $F_n(\zeta,z) = F_{n-1}(\zeta,f_n(\zeta,z))$ and
$F_{1,n}(\zeta,z) = f_{1,n}(\zeta,z)$, $F_{p,n}(\zeta,z) = F_{p-1,n}(\zeta,f_{p,n}(\zeta,z))$

(2) $G_1(\zeta,z) = f_1(\zeta,z)$, $G_n(\zeta,z) = f_n(\zeta,G_{n-1}(\zeta,z))$ and
$G_{1,n}(\zeta,z) = f_{1,n}(\zeta,z)$, $G_{p,n}(\zeta,z) = f_{p,n}(\zeta,G_{p-1,n}(\zeta,z))$

Then: (1) $\lim_{n\to\infty} F_{n,n}(\zeta,z) = \lim_{n\to\infty} F_n(\zeta,z) = \lambda(\zeta)$, analytic on S , *and*

(2) $\lim_{n\to\infty} G_{n,n}(\zeta,z) = \alpha(\zeta) = \lim_{k\to\infty}\alpha_k(\zeta)$, where $\alpha_k(\zeta) = f_k(\zeta,\alpha_k(\zeta))$,
analytic on S.

Proof: The previously cited *Lorentzen Theorem* and its extension by Gill, and Theorem 3.3 . The *Stieltjes-Vitali Theorem* [9] confirms the analyticity of the limit functions. ||

Example: *Tannery Continued Fraction*

Set $f_{k,n}(\zeta,z)=\dfrac{\zeta}{C+\delta(k,n)+z}$, where $C\geq 3$, $0\leq\delta(k,n)\leq 1$, $\lim\limits_{n\to\infty}\delta(k,n)=0$

Let $S=D=(|z|<1)$ and $\Omega=(|z|\leq\frac{3}{4})$. Thus $\left|f_{k,n}-f_k\right|<\delta(k,n)\cdot\dfrac{1}{(C-1)^2}$.

Then $F_{n,n}(\zeta,z)=\dfrac{\zeta}{C+\delta(1,n)}\;+\;\dfrac{\zeta}{C+\delta(2,n)}+\cdots+\dfrac{\zeta}{C+\delta(n,n)+z}$ and

$\lim\limits_{n\to\infty}F_{n,n}(\zeta,z)=\dfrac{\zeta}{C+}\dfrac{\zeta}{C+}\cdots=\alpha(\zeta)=\dfrac{1}{2}\left(\sqrt{C^2+4\zeta}-C\right)$.

Constructing Tannery Series from analytic functions . . .

Theorem 5.8 : Let f be a function analytic on D = (|z|<1), with f(0) = 0 , and |f(z)|<R. Let $\alpha=\alpha(k,n)$ and $\beta=\beta(k,n)$, $k\leq n$, be real and imaginary parts of points within D. If $\sum\limits_{k=1}^{n}|\alpha|+\sum\limits_{k=1}^{n}|\beta|\leq M$ for all n, then $S(n)=\sum\limits_{k=1}^{n}\left|f(\alpha(k,n)+i\beta(k,n))\right|\leq MR$ and there exists at least one subsequence $\{n_j\}$ such that $\{S(n_j)\}$ converges.

Proof: A simple application of *Schwarz's Lemma* suffices. ||

Of possible value is the following

Corollary: Suppose $\sum\limits_{k=1}^{n}|\alpha|+\sum\limits_{k=1}^{n}|\beta|\;\to\;0$ as $n\to\infty$. Then, for <u>any</u> function f defined as above, $\sum\limits_{k=1}^{n}\left|f(\alpha(k,n)+i\beta(k,n))\right|\;\to\;0$ as $n\to\infty$.

Here's an easy application:

Exercise: Use the function $f(z)=\dfrac{e^z-z-1}{z}$ in the corollary above to prove that $S(n) = (e^{\frac{1}{n}} - 1) + 2(e^{\frac{1}{2n}} - 1) + \cdots + n(e^{\frac{1}{n \cdot n}} - 1) \;\to\; 1$.

Evaluation of Tannery Series Not Satisfying Tannery's Theorem

Consider $\quad S_n = \sum_{k=1}^{n} a_k(n) \quad$ with $\quad a_k(n) \to a_k \quad$ but $\quad \lim_{n\to\infty} S_n \neq \sum_{k=1}^{\infty} a_k$.

The simplest interesting example of such a convergent TS is

$$\phi_n = \sum_{k=1}^{n} \frac{1}{n}\phi\left(\frac{k}{n}\right) \;\to\; \int_0^1 \phi(x)dx .$$

Suppose one wishes to evaluate

$$\psi_n = \sum_{k=1}^{n} \frac{kn + k^2 + n^2 + 1}{n(n^2 + k)} = \sum_{k=1}^{n} \frac{1}{n}\ \psi(k,n)\ ,\ \psi(k,n) = \frac{kn + k^2 + n^2 + 1}{n^2 + k} .$$

Then $\psi(k,n) = \dfrac{\left(k/n\right) + \left(k/n\right)^2 + 1 + 1/n^2}{1 + \left(k/n\right)\cdot 1/n} \approx \phi\left(\dfrac{k}{n}\right)$ where $\phi(\mathrm{x}) = x^2 + x + 1$, and one might suspect that $\psi_n \to \dfrac{11}{6}$. Indeed, this is the case, as is seen in the following simple theorem:

Theorem 5.9 : Given $\quad \psi_n = \dfrac{1}{n}\sum_{k=1}^{n} \psi(k,n)$,

suppose there exists an integrable function $\phi(x)$ with

$$I = \int_0^1 \phi(x)dx \quad \text{and} \quad \Delta(k,n) = \left|\phi\left(\frac{k}{n}\right) - \psi(k,n)\right| \le \varepsilon_n \cdot R\ ,\ \varepsilon_n \to 0 .$$

Then $\qquad \lim_{n\to\infty} \psi_n = I$.

Proof: $\quad \left|\phi_n - \psi_n\right| \le \varepsilon_n \cdot n \cdot \dfrac{R}{n} \to 0 \qquad$ ||

Example: $\psi_n = \sum_{k=1}^{n} \frac{Sin\left(\frac{k^2+k}{n^2+1}\right)}{n}$. Thus $\Delta(k,n) = \left|Cos(\eta_{k,n})\right| \cdot \left|\frac{k^2}{n^2} - \frac{k^2+k}{n^2+1}\right| \le \cdots \le \frac{1}{n} \cdot 2$.

Hence $\lim_{n\to\infty} \psi_n = \int_0^1 Sin(x^2)dx$.

Here's a curiosity couched as a problem: Partition the interval [0,1] into uneven, increasingly lengthy subintervals, going from 0 to 1, to obtain $\int_0^1 (x^2+1)dx = \lim_{n\to\infty} \sum_{k=1}^{n} T_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2k^3(k+1)^2 + 2kn^2(n+1)^2}{n^3(n+1)^3}$. Then manipulate T_n so as to obtain $\lim_{n\to\infty} \sum_{k=1}^{n} T_n = \int_0^1 \left(2x^5 + 2x\right)dx$!

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