Tannery's Theorem

John Gill

Forward iteration ($f_1 \circ f_2 \circ \cdots \circ f_n(z)$) and *backward iteration* ($f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$) are variations on simple iteration, and their convergence behaviors may reflect that of simple iteration of a *contraction mapping* described by Henrici [ ]. Investigations of the more complicated structures $f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(z)$ and $f_{n,n} \circ f_{n-1,n} \circ \cdots \circ f_{1,n}(z)$ lead to an extension of the classical Tannery's Theorem [1].

Tannery's Theorem [1] provides sufficient conditions on the series-like expression $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$ that it converge to the limit of the series $a_1 + a_2 + \cdots$, when $\lim_{n\to\infty} a_k(n) = a_k$ for each k. In fact, the original theorem provided this result for a more general series-like expansion , $S(p,n) = a_1(n) + a_2(n) + \cdots + a_p(n)$, where it is understood that p tends steadily to infinity with n. <u>In this and subsequent notes p will be taken to be n</u>.

Tannery's Theorem (series): Suppose that $S(n) = a_1(n) + a_2(n) + \cdots + a_n(n)$, where $\lim_{n\to\infty} a_k(n) = a_k$ for each k. Furthermore, assume $|a_k(n)| \le M_k$ with $\sum M_k < \infty$.

Then $\lim_{n\to\infty} S(n) = a_1 + a_2 + \cdots$, convergent.

Setting $f_{k,n}(z) = a_k(n) + z$, then

$$S(n) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(0), \text{ or } \quad S(n) = f_{n,n} \circ f_{n-1,n} \circ \cdots \circ f_{1,n}(0) .$$

Comment: The hypotheses can be weakened – see Tannery's Theorem Trivia 1

The classical Tannery theory can easily be extended to infinite products:

Tannery's theorem (products): Suppose that $P(n) = \prod_{k=1}^{n} (1 + a_k(n))$. If $\lim_{n\to\infty} a_k(n) = a_k$, and $|a_k(n)| \le M_k$ with $\sum M_k < \infty$, then $\lim_{n\to\infty} P(n) = \prod_{k=1}^{\infty} (1 + a_k)$.

However, the original theorem is less adaptable to more exotic expansions like continued fractions:

$$C(n) = \frac{a_1(n)}{1+}\ \frac{a_2(n)}{1+}\ \cdots\ \frac{a_n(n)}{1},$$

which can be expressed as

$$C(n) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{n,n}(0), \quad \text{with} \quad f_{k,n}(z) = \frac{a_k(n)}{1+z} \ .$$

Nevertheless, a unifying principle applicable to a variety of expansions exists if we restrict our attention to scenarios in which all functions $f_n(z)$ and $f_{k,n}$ map a simply-connected domain into a compact subset of itself. See the notes following this page.

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A Continuous Analog of Tannery's Theorem

A continuous analog of the theorem described above is the following:

Theorem [2]: [Let $\{f_n(x)\}$ be a sequence of functions continuous on R]. Suppose $\lim_{n\to\infty} f_n(x) = g(x)$ uniformly in any fixed interval, and there exists a positive function M(x) where $|f_n(x)| \le M(x)$ and $\int_a^\infty M(x)dx$ converges.

Then

$$\lim_{n\to\infty} \int_a^n f_n(x)dx = \int_a^\infty g(x)dx \ .$$

References:

[1] J. Tannery, *Fonctions d'une Variable*, Sec. 183
[2] T. Bromwich, *Introduction to the Theory of Infinite Series*, 2nd Ed, 1926