

Random Compositions of Analytic Functions

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Given a sequence of functions $\{f_n\}$ analytic on a domain D , with the range of each function in D , what can one say about the convergence behavior of a *random pattern of compositions* ?

For example: $\cdots \circ f_{k(2)} \circ f_{k(1)} \circ f_{k(3)} \circ \cdots$

where compositions may be both on the left (*backward iteration* or *outer composition*) and on the right (*forward iteration* or *inner composition*).

The following two theorems provide an easy answer :

Theorem 1. (inner composition); Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each n , $f_n(D) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ converges uniformly in D to a constant function $F(z) = \lambda$.

Theorem 2. (outer composition); Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D . Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ for all n . Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. If the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in Ω converge to a number α , then $G_n(z) \rightarrow \alpha$ uniformly on the closure of D .

(Note: That the hypotheses cannot be significantly reduced is shown by the example $g_n(z) = -.5$ for n odd and $g_n(z) = .5$ for n even, in the unit disk ($|z| < 1$). It is not essential that $g_n \rightarrow g$, although that is usually the case. If $g_n \rightarrow g$, then $\alpha_n \rightarrow \alpha$.)

Therefore, we conclude

Theorem 3. (random composition): Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D . Suppose there exists a compact set $\Omega \subset D$ such that for each n , $f_n(D) \subset \Omega$. Suppose $\{k(n)\}$ is a random reordering of the natural numbers, and $\{s(n)\}$ is a random sequence of 0s and 1s.

$$\text{Set } \mu_n(z) = \begin{cases} f_{k(n)}(z) & \text{if } s(n) = 0 \\ z & \text{if } s(n) = 1 \end{cases} \quad \text{and} \quad \eta_n(z) = \begin{cases} f_{k(n)}(z) & \text{if } s(n) = 1 \\ z & \text{if } s(n) = 0 \end{cases}$$

$$\text{Next set } \Psi_1(z) = f_{k(1)} \quad \text{and} \quad \Psi_n(z) = \mu_n \circ \Psi_{n-1} \circ \eta_n(z)$$

Then $\{\Psi_n(z)\}$ converges uniformly in D to a constant function $\Psi(z) = \lambda$, provided the sequence of *fixed points* $\{\alpha_n\}$ of the $\{f_n\}$ in Ω converge.

Comment: "random" of course implies an infinite *outer* composition, thus

$$\lim_{n \rightarrow \infty} \Psi_n(z) = \lambda = \alpha = \lim_{n \rightarrow \infty} \alpha_n$$

Even in the "impossible" scenario in which, from some point on, all compositions are on the right, convergence to a constant function still occurs, by Theorem 1.