# Convergence of Infinite Compositions of Complex Functions

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ABSTRACT: *Inner Composition* of analytic functions ( $f_1 \circ f_2 \circ \cdots \circ f_n(z)$ ) and *Outer Composition* of analytic functions ( $f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$ ) are variations on simple iteration, and their convergence behaviors as n becomes infinite may reflect that of simple iterations of *contraction mappings**. Several theorems are combined to give a summary of work in this area. In addition, recent results by the author and others provide convergence information about such compositions that involve functions that are not contractive, and in some cases, neither analytic nor meromorphic.
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* $\phi$ defined on a simply-connected domain S with $\phi(S) \subset \Omega \subset S$ , $\Omega$ compact.

# Table of Contents

**Historical Background**

Among the previously investigated classes of special functions with regard to both inner and outer composition are *linear fractional transformations* (LFTs) :

$f_n(z)=\dfrac{a_n z+b_n}{c_n z+d_n}$ , normalized by $D_n = a_n d_n - b_n c_n = 1$. If the LFT has two fixed points $\alpha_n \neq \beta_n$ it may be written in *multiplier format*: $\dfrac{f_n(z)-\alpha_n}{f_n(z)-\beta_n} = K_n \cdot \dfrac{z-\alpha_n}{z-\beta_n}$. Thus LFTs fall into four distinct types:

Hyperbolic if $0 < K_n < 1$

Elliptic if $K_n = e^{i\theta_n}$, $\theta_n \neq 0 (\mathrm{mod}\, 2\pi)$

Loxodromic if $K_n = |K_n| e^{i\theta_n}$, $0 < |K_n| < 1$, and $\theta_n \neq 0 (\mathrm{mod}\, 2\pi)$

Parabolic if $\alpha_n = \beta_n$

Both continued fractions and series may be perceived as infinite compositions of certain classes of LFTs; and in the case of continued fractions, especially the analytic theory of continued fractions, many conclusions derive from studies using this approach. The following theorems, however, touch upon the highlights of investigations into the pure convergence behavior of composition sequences of general LFTs:

Set $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ and $G_n(z) = f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$ where $\lim_{n\to\infty} f_n = f$ .

Theorem [9] If $\prod a_n$ , $\sum b_n$ , $\sum c_n$ , and $\prod d_n$ all converge absolutely, then $F_n \to F$, a LFT.

Theorem [9] If $F_n \to F$, a LFT, then $\lim_{n\to\infty} f_n = f = z$.

Theorem [10] If $\lim_{n\to\infty} f_n = f$ and all functions are *hyperbolic* or *loxodromic*, then $F_n \to C_0$ for all $z \neq \beta = \mathrm{Lim}\, \beta_n$.

Theorem [10] If $\{f_n\}$ are *elliptic* and $\lim_{n\to\infty} f_n = f$ is *elliptic*, and $\sum |\beta_n - \beta_{n-1}| < \infty$, $\sum |\alpha_n - \alpha_{n-1}| < \infty$, and $\sum |\beta_n - \beta + \alpha - \alpha_n| < \infty$, then $\{F_n(z)\}$ diverges for all $z \neq \alpha, \beta$, with $F_n(\alpha) \to \alpha^*$ and $F_n(\beta) \to \beta^*$ , $\alpha^* \neq \beta^*$.

Theorem [2] Suppose $\{f_n\}$ are LFTs converging to a *parabolic* LFT , f , with $\alpha_n \to \alpha$ finite. If $\sum n|\beta_{n+1} - \beta_n| < \infty$ and $\sum |\alpha_n - \beta_n| < \infty$, then $\mathrm{F_n} \to \mathrm{C_0}$ for all z.

Theorem [2] Given $\{f_n\}$ LFTs with $\lim_{n\to\infty} f_n = f$ *elliptic* with fixed points $\alpha$ and $\beta$.

(i) If $\sum |\alpha_n - \alpha_{n-1}| < \infty$ , $\sum |\beta_n - \beta_{n-1}| < \infty$ , and $\prod K_n$ diverges to 0, then $\mathrm{F_n} \to \mathrm{C_0}$ for all $z \neq \beta$.

(ii) If $\sum |\alpha_n - \alpha_{n-1}| < \infty$ , $\sum |\beta_n - \beta_{n-1}| < \infty$ , and $\prod K_n$ converges, then $\{\mathrm{F_n(z)}\}$ diverges for $z \neq \alpha, \beta$ while $F_n(\alpha) \to \alpha^*$ and $F_n(\beta) \to \beta^*$

Theorem [11] Suppose

(i) $\mathrm{K_n} = |\mathrm{K_n}| \mathrm{e}^{i\theta_n}$, $0 < |\mathrm{K_n}| < 1$, $\mathrm{K_n} \to 1$ $(\mathrm{f_n} \to \mathrm{z})$ or $\mathrm{K_n} \to 0$ $(\mathrm{f_n} \to \alpha)$

(ii) $\alpha_n \to \alpha$ , $\beta_n \to \beta$ , $\alpha \neq \beta$

(iii) $\sum |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum |\beta_n - \beta_{n-1}| < \infty$

If $\prod K_n = 0$ then $\mathrm{F_n} \to \mathrm{C_0}$ for $z \neq \beta$ and $\mathrm{G_n} \to \alpha$ (except perhaps at one point)

If $\prod K_n$ converges absolutely then both $\mathrm{F_n}$ and $\mathrm{G_n}$ converge to LFTs.

These theorems are generalizations of the simple and elegant convergence and divergence behavior of iterations of single LFTs [12].

The remainder of this article is devoted to exploring the convergence behavior of infinite inner and outer compositions of more general functions. Although the condition $\lim_{n\to\infty} f_n = f$ , sometimes called *limit periodic*, plays a significant role in the theorems listed above, it is not required in most of what follows.

## 1. Contractive Functions

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Theorem 1.1(Gill) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each n, $f_n(D) \subset \Omega$. Thus for each n, there exists a unique $\alpha_n = f_n(\alpha_n) \in \Omega$. Define $\{k(n)\}$ to be an ordering of the positive integers and $\{s(n)\}$ an infinite sequence of 0s and 1s.

Set $\mu_n(z) = \begin{cases} f_{k(n)}(z) & \text{if } s(n)=0 \\ z & \text{if } s(n)=1 \end{cases}$ and $\eta_n(z) = \begin{cases} f_{k(n)}(z) & \text{if } s(n)=1 \\ z & \text{if } s(n)=0 \end{cases}$

Next, set $\Psi_1(z) = f_{k(1)}$ and $\Psi_n(z) = \mu_n \circ \Psi_{n-1} \circ \eta_n(z)$

The convergence behaviors of $\{\Psi_n(z)\}$ under all possible choices of $\{k(n)\}$ and $\{s(n)\}$ follows:

**(1)** k(n) = any ordering of $\{n\}$ <u>and</u> s(n) = 0 for all n: <u>*Outer Composition*</u>
Then $\Psi_n(z) = f_{k(n)} \circ f_{k(n-1)} \circ \cdots \circ f_{k(1)}(z)$ and
$\Psi_n(z) \to \alpha$ if and only if the sequence of fixed points $\{\alpha_n\}$ of $\{f_n\}$ converge to $\alpha$.
Convergence is uniform on D.

**(2)** k(n) = any ordering of $\{n\}$ <u>and</u> s(n) =1 for all n: <u>*Inner Composition*</u>

Then $\Psi_n(z) = f_{k(1)} \circ f_{k(2)} \circ \cdots \circ f_{k(n)}(z)$ and
$\Psi_n(z) \to \lambda(k)$ , where $\lambda(k)$ is a function of the sequence $\{k(n)\}$,
and a constant for each such sequence. Convergence is uniform on D.

**(3)** k(n) = any ordering of $\{n\}$ <u>and</u> $\{s(n)\}$ is any (random) sequence of 0s and 1s:

Then, assuming there are an infinite number of 0s in the sequence $\{s(n)\}$,
$\Psi_n(z) \to \alpha$ if and only if the sequence of fixed points $\{\alpha_n\}$ of $\{f_n\}$ converge to $\alpha$.
If there are only a finite number of 0s,

$\Psi_n(z) \to \lambda$ , where $\lambda$ depends upon the sequence $\{k(n)\}$

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*Discussion of the theorem*:

Inner Composition

We begin with a well-known and very basic result:

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Theorem 1.2 (Henrici [1], 1974) Let f be analytic in a simply-connected region S and continuous on the closure S' of S. Suppose f(S') is a bounded set contained in S. Then $f^n(z) = f \circ f \circ \cdots \circ f(z) \to \alpha$ , the *attractive fixed point* of f in S, for all z in S'.

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This result can be extended to *forward iteration* (or *inner composition*) involving a sequence of functions. The author began a study of infinite compositions of linear fractional transformations in [2].

Later, in the 1980s, the author initiated a study of extensions of *limit-periodic continued fractions*, in which sequences

$$F_n(z) = F_{n-1}(f_n(z)) = f_1 \circ \cdots \circ f_n(z) \text{ , with } f_n \to f$$

involving functions other than those giving rise to continued fractions (viz., $f_n(z) = \frac{a_n}{b_n + z}$, a Mobius Transformation) were investigated:

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Theorem 1.3 (Gill [3], 1988) Suppose there exist a sequence of regions $\{D_n\}$ and a sequence of functions $\{f_n\}$ analytic on those respective regions, such that $f_n(D_n) \subset D_{n-1}$ ,and there exists a region $D \subset \cap D_n$ with $f_n \to f$ uniformly on compact subsets of D. Furthermore, suppose there exists
$\alpha \in \overline{D}$ such that $|f(z) - \alpha| < |z - \alpha|$ for all $z \in$ Boundary$(\overline{D})$.
Then there exists $\lambda \in D_1$ such that $\lim_{n\to\infty} F_n(z_n) = \lambda \quad \forall \{z_n\}$ in $\overline{D}$.

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Lorentzen generalized this idea in the following (sometimes called the Lorentzen-Gill Theorem):

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Theorem 1.4 (Lorentzen [4], 1990 ) Let $\{f_n\}$ be a sequence of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each n, $f_n(D) \subset \Omega$. Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ converges uniformly in D to a constant function F(z) = $\lambda$.

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*Example*: The simple continued fraction $\dfrac{a_1 z}{1} \; \dfrac{a_2 z}{1} \; + \cdots$ is generated by an inner composition in which

$$f_n(z,w) = \frac{a_n z}{1+w},\ F_1(z,w) = f_1(z,w) \ \text{and}\ F_n(z,w) = F_{n-1}(z, f_n(z,w)).$$ Suppose $|z|<1$ and $|w|<R<1$. Then $|a_n| < \rho R(1-R) \Rightarrow |f_n(z,w)| < \rho R$, $0<\rho<1$. For R=1/2, for instance, $F_n(z,w) \to \phi(z)$ analytic in ($|z|<1$).

*Example*: The exponential tower generated by $f_n(z,w) = \frac{1}{n+2} e^{zw}$, $|z|<1$, $|w|<R$ : $F_n(z,w) = F_{n-1}(z, f_n(z,w))$ is convergent, for instance for R=1, to a limit function $\phi(z)$ analytic in ($|z|<1$).

*Example*: Set $f_n(z,w) = \int_0^z \psi_n(t,w)dt$ where $|z|<1$, $|w|<1$, $|\psi_n(t,w)| < \rho < 1$, and generate $F_n(z,w) = F_{n-1}(z, f_n(z,w))$ to obtain a function $\phi(z)$ analytic in ($|z|<1$). For example, set

$$f_n(z,w) = \int_0^z \frac{e^t}{n+4+tw} dt \quad , \text{ giving } F_n(z,w) = \int_0^z \frac{e^t}{5 + t\displaystyle\int_0^z \frac{e^t}{6+\ddots} dt} dt,$$ which converges to a function $\phi(z)$ analytic in ($|z|<1$).

Additional Related Result:

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Theorem 1.5 (Gill, [5],1992) Let $\{f_{k,n}\}$, $1 \le k \le n$ be a family of functions analytic on a simply-connected domain D. Suppose there exists a compact set $\Omega \subset D$ such that for each k and n, $f_{k,n}(D) \subset \Omega$ and, in addition, $\lim_{n\to\infty} f_{k,n}(z) = f_k(z)$ uniformly on D for each k. Then , with $F_{p,n}(z) = f_{1,n} \circ f_{2,n} \circ \cdots \circ f_{p,n}(z)$ ,

$$F_{n,n}(z) \to \lambda \text{ , a constant function , as } n \to \infty \text{, uniformly on D.}$$

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**Comment**: The condition $\lim_{n\to\infty} f_{k,n}(z) = f_k(z)$, if discarded, allows the possibility of divergence by oscillation: viz.,

$$f_{1,n}(z) = \begin{cases} .5 \text{ if n is odd} \\ -.5 \text{ if n is even} \end{cases} \text{, otherwise } f_{k,n}(z) \equiv \frac{z}{2}, \text{ on } S = (|z| < 1) \text{ .}$$

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<u>Outer Composition</u>

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Theorem 1.6 (Gill [6], 1991) Let $\{g_n\}$ be a sequence of functions analytic on a simply-connected domain D and continuous on the closure of D. Suppose there exists a compact set $\Omega \subset D$ such that $g_n(D) \subset \Omega$ for all n. Define $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$. Then $G_n(z) \to \alpha$ uniformly on the closure of D *if and only if* the sequence of *fixed points* $\{\alpha_n\}$ of the $\{g_n\}$ in $\Omega$ converge to the number $\alpha$.

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*comments*: The existence of the $\{\alpha_n\}$ is guaranteed by Theorem 1. Note the simple counter-example $g_n(z) = -.5$ for n odd and $g_n(z) = .5$ for n even, in the unit disk (|z|<1). It is not essential that $g_n \to g$, although that is usually the case. If $g_n \to g$, then $\alpha_n \to \alpha$.

The proof of the assertion that $G_n(z) \to \alpha$ is found in the paper cited above.

However, the proof of the converse (*Gill*, 2010) – that the sequence of (*attractive*) fixed points converge to a common limit – is sketched here: For clarity, assume D = (|z|<1) and $\lim_{n\to\infty} G_n(z) = \lim_{n\to\infty} g_n \circ g_{n-1} \circ \cdots \circ g_1(z) = \alpha = 0$ (uniformly) for all z in D. The Riemann Mapping theorem and Mobius functions can be employed to extend these results.

(1) Schwarz' Lemma can be used to prove the following:
There exist $\{\rho_n\}$ $\forall n$ such that $|g_n(z) - \alpha_n| < \rho|z - \alpha_n|$ where $0 \le \rho_n < 1$ and $\operatorname*{Sup}_n \rho_n = \rho < 1$.

(2) Suppose the fixed points $\alpha_n$ do not converge to $\alpha$=0. Then there exists a sequence

$$\{\alpha_{n_j}\} \text{ and } \lambda, \text{ with } r = |\lambda| > 0 \text{ such that } |\alpha_{n_j} - \lambda| < \delta = \frac{r(1-\rho)}{4(1+\rho)} \text{ for all } n_j \text{ sufficiently large}.$$

(3) There exists N such that n>N implies $|w_n| < \varepsilon = \dfrac{r(1-\rho)}{4(1+\rho)}$, where $w_n = G_n(z)$

(4) From (1), above

$$\begin{aligned} |G_{n_j}(z) - \alpha_{n_j}| = |w_{n_j} - \alpha_{n_j}| &< \rho|w_{n_j-1} - \alpha_{n_j}| \\ &< \rho|w_{n_j-1}| + \rho|\alpha_{n_j}| \\ &< \rho\varepsilon + \rho(r+\delta) \end{aligned}$$

from which one obtains

$$|w_{n_j}| > r - \delta - \rho(r + \varepsilon + \delta) > \varepsilon \quad (\rightarrow\leftarrow). \,\|$$

Outer composition can be used to determine the solution of certain *fixed-point equations*.

*Example*: let $G(z) = \cfrac{\frac{e^z}{4}}{3+z} + \cfrac{\frac{e^z}{8}}{3+z} + \cfrac{\frac{e^z}{12}}{3+z} + \cdots$ , where $|z| \le 1$. We solve the *continued fraction equation* $G(\alpha) = \alpha$ in the following way: Set $t_n(\xi) = \dfrac{e^z/4n}{3+z+\xi}$ ; let $g_n(z) = t_1 \circ t_2 \circ \cdots \circ t_n(0)$.
Now calculate $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$ , starting with z = 1. One obtains $\alpha = .087118118\cdots$ to ten decimal places after ten iterations.

Certain *integral equations* may be solved numerically using outer compositions.

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Theorem 1.7 (Gill, 2011) Given $\varphi(\zeta,t)$ analytic in $D=\left(|\zeta|<R\right)$ for each $t\in[0,1]$ and continuous in t, set $g_n(\zeta)=\frac{1}{n}\left(\varphi\left(\zeta,\frac{1}{n}\right)+\varphi\left(\zeta,\frac{2}{n}\right)+\cdots+\varphi\left(\zeta,\frac{n}{n}\right)\right)$ and $G_n(\zeta)=g_n\left(G_{n-1}(\zeta)\right)$ , $G_1(\zeta)=g_1(\zeta)$ . Suppose $|\varphi(\zeta,t)|\le r<R$ for $\zeta\in D$. Then

$\zeta=\int_0^1\varphi(\zeta,t)dt$ has a unique solution, $\alpha$, in D, with $\lim_{n\to\infty}G_n(\zeta)=\alpha$.

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*Sketch of Proof*: $|g_n(\zeta)|\le r<R$ satisfies the hypotheses of theorem 1.6, with $g_n(\zeta)\to g(\zeta)=\int_0^1\varphi(\zeta,t)dt \Rightarrow \alpha_n\to\alpha$ , so that $\lim_{n\to\infty}G_n(\zeta)=\alpha$.

And $G_n(\zeta)=\frac{1}{n}\left(\varphi\left(G_{n-1}(\zeta),\frac{1}{n}\right)+\varphi\left(G_{n-1}(\zeta),\frac{2}{n}\right)+\cdots+\varphi\left(G_{n-1}(\zeta),\frac{n}{n}\right)\right)$ implies

$$\alpha=\int_0^1\varphi(\alpha,t)dt \qquad ||$$

*Example*: $\zeta=\int_0^1\frac{e^{i\zeta t^2}}{2}dt$ , $\alpha=(.47712\cdots)+i(.07483\cdots)$

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Theorem 1.7b (Gill, 2011) Suppose $\varphi(\zeta,t)$ is analytic for $S=\left(|\zeta|<R\right)$ and $t\ge0$, with $|\varphi(\zeta,t)|<M_n$ for $n\text{-}1\le t<n$ and $\sum_{k=1}^{\infty}M_k<r<R$. Set

$$g_n(\zeta)=\delta_n\left(\varphi(\zeta,\delta_n)+\varphi(\zeta,2\delta_n)+\cdots+\varphi\left(\zeta,(n/\delta_n)\cdot\delta_n\right)\right)$$

$\delta_n=\frac{1}{\sigma(n)}\to0$ , $\sigma(n)$ an integer , $G_1(\zeta)=g_1(\zeta)$ and $\mathrm{G}_n(\zeta)=g_n\left(G_{n-1}(\zeta)\right)$.

Then $S(\zeta)=\int_o^\infty\varphi(\zeta,t)\,dt$ has a unique fixed point $\alpha=S(\alpha)=\lim_{n\to\infty}G_n(\zeta)$.

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*Example*: $S(\zeta)=\int_0^\infty \frac{dt}{4e^{(\zeta+2)t^2}+i}$ implies $\alpha=\int_0^\infty \frac{dt}{4e^{(\alpha+2)t^2}+i}\approx .1465\cdots+\ i(-.0248\cdots)$

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Theorem 1.7c (Gill, 2011) Given $\varphi(\zeta,z,t)$ analytic in $D=(|\omega|<R)$ for both z and $\zeta$, for each $t\in[0,1]$ and continuous in t. Set

$$g_n(\zeta,z)=\frac{1}{n}\left(\varphi\left(\zeta,z,\frac{1}{n}\right)+\varphi\left(\zeta,z,\frac{2}{n}\right)+\cdots+\varphi\left(\zeta,z,\frac{n}{n}\right)\right)$$

and $G_n(\zeta)=g_n(\zeta,G_{n-1}(\zeta))$, $G_1(\zeta)=g_1(\zeta,z)$. Suppose $|\varphi(\zeta,z,t)|\le r<R$. Then

$\alpha(\zeta)=\int_0^1 \varphi(\zeta,\alpha(\zeta),t)dt$ has a unique solution, $\alpha(\zeta)$, in D, with $\lim_{n\to\infty} G_n(\zeta)=\alpha(\zeta)$.

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*Example*: $\alpha(\zeta)=\int_0^1 \frac{e^{\zeta z t^2}}{4}\,dt=\int_0^1 \frac{e^{\zeta\alpha(\zeta)t^2}}{4}\,dt$ with

$\alpha(.1-.3i)=.2518\cdots-(.00654\cdots)i$ ,
$\alpha(0+.2i)=.2498\cdots+(.00423\cdots0)i$ ,
$\alpha(.6+.1i)=.2640\cdots+(.00261\cdots)i$

Additional Related Result:

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Theorem 1.8 (Gill [7], 2010) Suppose $\{g_{k,n}\}$, with $k \le n$, is a family of functions analytic on a simply-connected domain D and continuous on its closure, with $g_{k,n}(D) \subset \Omega$, a compact subset of D, for all k and n.
Then

$$G_{n,n}(z) = g_{n,n} \circ g_{n-1,n} \circ \cdots \circ g_{1,n}(z) \to \alpha \text{ uniformly on the closure of D}$$

*if and only if*

the sequence of fixed points $\{\alpha_{k,n}\}$ of $\{g_{k,n}\}$ converge* to $\alpha$.

(*Comments*: When $\lim_{n\to\infty} g_{k,n}(z) = g_k(z)$, for each value of k, both sequences converge to the limit described in theorem 2. * For $\varepsilon>0$ $\exists$ $N=N(\varepsilon)$ $\ni$ $N<k \le n$ $\Rightarrow$ $|\alpha_{k,n} - \alpha| < \varepsilon$)

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Simple Examples of Compositions of Infinite Expansions:

*Example*: $\phi_n(z) = a_{0,n} + a_{1,n}z + a_{2,n}z^2 + \cdots$ defined on $D = (|z|<1)$, with $\sum_{k=0}^{\infty} |a_{k,n}| \le r < 1$ implies the $\{\phi_n\}$ are uniformly contractive on D.
Hence $F_n(z) = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n(z) \to \lambda$ uniformly on compact subsets of D.

*Example*: $t_{k,n}(z,\zeta) = \dfrac{a_{k,n}z}{1+\zeta}$ with $|\zeta| \le r < 1$, $|a_{k,n}| \le C$, $|z| < R$ implies

$$|t_{k,n}| < \frac{CR}{1-r} = \rho < r \quad \text{if} \quad R < \frac{r(1-r)}{C}.$$

Then

$$T_{k,n}(z,\zeta) = T_{k-1,n}(z, t_{k,n}(z,\zeta)) \to \phi_n(z) \text{ for } |z| < R.$$

We have $\phi_n(z) = \dfrac{a_{1,n}z}{1} \underset{+}{} \dfrac{a_{2,n}z}{1} \underset{+}{} \cdots$ with $|\phi_n| < \rho < R$ provided $C < 1-r$. Thus the $\{\phi_n\}$ are uniformly contractive on (|z|<R). It follows that $F_n(z) = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n(z) \to \lambda$.

## 2. Non-Contractive Functions

**Inner Compositions**

Among other results, Shota Kojima proved the following, using an inventive and appropriate notation: $\mathop{\mathfrak{R}}_{n=k}^{N} f_n(z) = f_k \circ f_{k+1} \circ \cdots \circ f_N(z)$

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Theorem 2.1(Kojima [8], 2010) Consider entire functions $f_n(z) = z + \sum_{r=2}^{\infty} c_{n,r} z^r$ with complex coefficients. Set $C_n = \sup_{r=2,3,4...} \left\{ \left| c_{n,r} \right|^{\frac{1}{r-1}} \right\}$. Then the convergence of the series $\sum_{n=1}^{\infty} C_n$ implies $f(z) = \mathop{\mathfrak{R}}_{n=1}^{\infty} f_n(z)$ exists and is entire.

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*Corollary* : (Kojima, 2010) Let $\{c_n\}$ be a sequence of positive real numbers such that $\sum c_n < \infty$. Define $g_n(x) = \mathop{\mathfrak{R}}_{r=1}^{n} \left( x + c_r x^2 \right) = \left( x + c_1 x^2 \right) \circ \left( x + c_2 x^2 \right) \circ \cdots \circ \left( x + c_n x^2 \right)$ and $F(x) = \mathop{\mathfrak{R}}_{n=1}^{\infty} \left( x + c_n x^2 \right)$. Then the sequence $\{g_n(x)\}$ is uniformly convergent on compact subsets of C, with $\lim_{n \to \infty} g_n(x) = F(x)$, an entire function.

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Kojima investigates the expansions of functions satisfying the functional equation

$$f(sz) = s\, f(z) + s\, f(z)^2 \quad \text{with complex s where } |s| > 1.$$

*Example* (*Kojima,* 2010): $\frac{1}{2}\left(e^{2z} - 1\right) = \mathop{\mathfrak{R}}_{n=1}^{\infty} \left( z + \frac{z^2}{2^n} \right)$

Comments About Kojima's Theorem and other Related Theory

For these infinite inner compositions to converge, there frequently is required either some sort of contractive property, as seen in Lorentzen's Theorem and my theorem above, or some sort of Lipshitz condition or uniform convergence:

$$|f_n(u) - f_n(v)| < \rho_n |u - v| \quad \text{or} \quad |f_n(z) - f(z)| < \rho_n |z| \quad \text{, or something similar.}$$

Speed of uniform convergence is often geometrical. Almost all theorems on this subject provide sufficient conditions for convergence of infinite compositions. Continued fraction theory is a case in point. The only general theorem that I am aware of that is both *sufficient and necessary* for convergence is theorem 1.6.

A general strategy is to deal with the "tail end" of the composition, applying the sorts of techniques mentioned above, showing the tail end converges. Then all that remains is the simple step of composing the front section – which is an analytic function – to the convergent tail section.

Kojima's Theorem provides a glimpse of the inner structures most easily developed in this regard. The following is a *sketch* of *a* proof of his result, using these conditions.

Start with the observation that each of the power series in the hypothesis is majorized by a geometric series:

$$f_n(z) = z + a_{n,2}z^2 + a_{n,3}z^3 + \cdots, \quad |z| \le R \quad \Rightarrow$$

$$\begin{aligned} |f_n(z)| &\le |z|\left(1 + |a_{n,2}||z| + |a_{n,3}||z|^2 + \cdots\right) \\ &\le |z|\left(1 + C_n|z| + C_n^2|z|^2 + \cdots\right) \\ &\le \frac{|z|}{1 - C_n|z|} \le \frac{R}{1 - C_nR} \end{aligned}$$

Let us choose and fix some n large enough to insure the following: $\sum_{j=0}^{\infty} C_{n+j} < \frac{1}{4R}$

Then, working with the tail end, we have

$$
(1) \qquad
\begin{aligned}
&|f_{n+m}(z)| \;\le\; \frac{R}{1-RC_{n+m}} \;<\; 2R \\
&|f_{n+m-1}\circ f_{n+m}(z)| \;\le\; \frac{|f_{n+m}(z)|}{1-C_{n+m-1}|f_{n+m}(z)|} < \frac{R}{1 - R\left(C_{n+m}+C_{n+m-1}\right)} \\
&. \\
&. \\
&. \\
&|f_n\circ f_{n+1}\circ\cdots\circ f_{n+m}(z)| \;<\; \frac{R}{1 - R\left(C_{n+m}+\cdots+C_n\right)} \;<\; 2R
\end{aligned}
$$

Using the same approach, look now at

$$
(2) \qquad |f_n(z)-z| \;\le\; C_n|z|^2\left(1+C_n|z|+C_n^2|z|^2+\cdots\right) \;\le\; \frac{C_n|z|^2}{1-C_n|z|}
$$

And finally , with $R_0 = 2R$ ,

$$
(3) \qquad
\begin{aligned}
|f_n(z)-f_n(\zeta)| \;&\le\; |z-\zeta|\left\{1+C_n(|z|+|\zeta|)+C_n{}^2(|z|^2+|z||\zeta|+|\zeta|^2)+\cdots\right\} \\
&\le\; |z-\zeta|\left\{1+2(C_nR_0)+3(C_nR_0)^2+4(C_nR_0)^3+\cdots\right\} \\
&\le\; |z-\zeta|\cdot\frac{1}{(1-R_0C_n)^2} \;=\; P_n\cdot|z-\zeta|
\end{aligned}
$$

Thus $|f_n(z)-f_n(\zeta)| < P_n\cdot|z-\zeta|$ , where it is easily confirmed that $\prod_{k=1}^{\infty} P_k$ converges since $\sum_1^{\infty} C_n$ converges. Write $\prod_{k=n}^{\infty} P_k < M(n)$ .

Set $F_{n,n+k}(z) = f_n\circ f_{n+1}\circ\cdots\circ f_{n+k}(z)$. To show $\{F_{n,n+m}(z)\}_{m=0}^{\infty}$ converges: prove it is a Cauchy sequence.

$$
\begin{aligned}
\left|F_{n,n+m}(z)-F_{n,n+m+p}(z)\right| &= \left|f_n(F_{n+1,n+m}(z))-f_n(F_{n+1,n+m+p}(z))\right| \\
&< P_n\cdot\left|f_{n+1}(F_{n+2,n+m}(z))-f_{n+1}(F_{n+2,n+m+p}(z))\right| \\
&< P_nP_{n+1}\left|f_{n+2}(F_{n+3,n+m}(z))-f_{n+2}(F_{n+3,n+m+p}(z))\right| \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&< \prod_{k=n}^{n+m}P_k\cdot\ \left|f_{n+m+1}(F_{n+m+2,n+m+p}(z))-z\right| \\
&< M(n)\cdot\ \left|f_{n+m+1}(F_{n+m+2,n+m+p}(z))-z\right| \\
&< M(n)\left(\left|f_{n+m+1}(F_{n+m+2,n+m+p}(z))-F_{n+m+2,n+m+p}(z)\right|+\left|f_{n+m+2}(F_{n+m+3,n+m+p}(z))-z\right|\right) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&< \frac{16M(n)R^2}{3}\cdot\sum_{k=m+n+1}^{\infty}\rho_k \;\to\; 0\text{, as } m\to\infty.
\end{aligned}
$$

Therefore,

$\lim_{k\to\infty}F_k(z)=\lim_{m\to\infty}F_{n-1}(F_{n,n+m}(z))=F_{n-1}(\lim_{m\to\infty}F_{n,n+m}(z))$ . Convergence is uniform on compact sets in the complex plane.ǁ

Generalizing this theorem a tad, I get :

---

Theorem 2.2 (Gill, 2010) Given a family of entire functions, $\{f_n\}$, suppose that for $|z|<R$ there exists an N=N(R) where n>N implies (with uniform convergence of the sums):

(i) $\left|f_n(z)-z\right|<\alpha_n(z)\left|z\right|$ , with $\alpha_n(z)\geq 0$ and $\sum_{n=1}^{\infty}\alpha_n(z)<\infty$

(ii) $\left|f_n(z)-f_n(\zeta)\right|<(1+\beta_n(z))\left|z-\zeta\right|$ , with $\beta_n(z)\geq 0$ and $\sum_{n=1}^{\infty}\beta_n(z)<\infty$

Then $\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$ exists, with convergence being uniform on compact sets in the complex plane.

---

*Sketch of Proof*:

First,

$$\left|f_{n+m}(z)\right| < \left(1+\beta_{n+m}\right)\cdot R$$
$$\vdots$$
$$\left|f_n \circ \cdots \circ f_{n+m}(z)\right| < \prod_{k=n}^{n+m}\left(1+\beta_k\right)\cdot R < M$$

Then,

$$\left|f_n \circ \cdots f_{n+m}(z) - f_n \circ \cdots f_{n+m}\left(f_{n+m+1} \circ \cdots \circ f_{n+m+p}(z)\right)\right|$$
$$< \prod_{k=n}^{n+m}\left(1+\beta_k\right)\cdot\left|z - f_{n+m+1} \circ \cdots \circ f_{n+m+p}(z)\right|$$
$$< \prod_{k=n}^{n+m}\left(1+\beta_k\right)\cdot\left\{M\rho_{n+m+1} + M\rho_{n+m+2} + \cdots\right\} \to 0 \text{ as } m \to \infty$$

Additional results related to Kojima's theorem. . .

---

Theorem 2.3 (Gill, 2010) Suppose $f_n(z) = a_1(n)z + a_2(n)z^2 + \cdots$ with $|z|<R$, $\left|a_k(n)\right| < {\rho_n}^k$, $\rho_n < 1$ and $\rho_n(\downarrow)0$. Then $\lim_{n\to\infty}\left(f_1 \circ \cdots \circ f_n(z)\right) = 0$ for $|z| < R$.

---

*Sketch of Proof*: Assume a large fixed value of n.

$$\left|f_j(z)\right| \le \rho_j |z| + \rho_j^2 |z|^2 + \cdots \le \rho_j \cdot \frac{|z|}{1-\rho_j |z|}$$

.
.
.

$$\left|f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z)\right| \le \frac{R \prod_{j=n}^{n+m} \rho_j}{1 - R \sum_{k=n}^{n+m} \left( \prod_{j=k}^{n+m} \rho_j \right)} = R_{n+m}$$

Where $\sum_{k=n}^{n+m} \left( \prod_{j=k}^{n+m} \rho_j \right) \le \rho_n + \rho_n^{\,2} + \cdots + \rho_n^{\,m+1} \le \frac{\rho_n}{1-\rho_n}$ and $\prod_{j=k}^{n+m} \rho_j \le \rho_n^{\,m+1}$.

Then $R_{n+m} \le \rho_n^{\,m+1} \cdot \frac{R(1-\rho_n)}{1-\rho_n(R+1)} \to 0$ as $m \to \infty$. ||

---

Theorem 2.4 (Gill, 2010) Suppose
$f_n(z) = a_0(n) + a_1(n)z + a_2(n)z^2 + \cdots + a_n(n)z^n$, where $|z| < R$,
$|a_0(n)| < \rho_n$ and $|a_k(n)| < \rho_n^{\,k}$, $\rho_n < 1$ and $\rho_n (\downarrow) 0$. Then
$\lim_{n\to\infty} \left( f_1 \circ \cdots \circ f_n(z) \right) = \lambda(z)$, analytic for $|z| < R$, with convergence being uniform on compact sets in C.

---

*Sketch of Proof*: Choose and fix n so large that $\rho_n < \frac{\delta}{R} < \frac{1}{R}$ and $\rho_n (\downarrow) 0$.

Then $\left|f_j(z)\right| < \rho_j \left( 1 + \frac{|z|}{1-\rho_j |z|} \right) < \rho_j \left( 1 + \frac{R}{1-\delta} \right) < R$ for large j.

This leads easily to

$\left|f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z)\right| < R$ for fixed large n and positive m. Next, as in Theorem 7,

$$\left|f_n(z)-f_n(\zeta)\right| \le \left|z-\zeta\right|\cdot\rho_n\left(1+2(R\rho_n)+3(R\rho_n)^2+4(R\rho_n)^3+\cdots\right)$$

$$\le \left|z-\zeta\right|\cdot\frac{\rho_n}{\left(1-R\rho_n\right)^2} < \frac{1}{2}\cdot\left|z-\zeta\right|$$

for $n$ sufficiently large.

Applying the Cauchy Condition for convergence,

$$\left|f_n\circ\cdots\circ f_{n+m}(z)-f_n\circ\cdots\circ f_{n+m}\circ f_{n+m+1}\circ\cdots\circ f_{n+m+p}(z)\right|$$

$$< \frac{1}{2^m}\left|z-f_{n+m+1}\circ\cdots\circ f_{n+m+p}(z)\right| \le \frac{1}{2^m}\left(\left|z\right|+\left|f_{n+m+1}\circ\cdots\circ f_{n+m+p}(z)\right|\right)$$

$$< \frac{R}{2^{m-1}}\to 0 \quad \text{as } m\to\infty.$$

||

---

Theorem 2.5 (Gill, 2011) Consider entire functions whose linear coefficients *approach* one:

Let $f_n(z) = a_n z + a_{2,n}z^2 + \cdots + a_{k,n}z^k + \cdots$ , where $a_n\to 1$,

and $\left|a_{k,n}\right|\le\rho_n^{k-1}$ with $\sum\rho_n<\infty$. Set $\varepsilon_n=\left|a_n-1\right|$ with $\sum\varepsilon_n<\infty$, and $\alpha_n=\left|a_n\right|$

Then $\lim_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$ , entire, with uniform convergence on compact sets in the complex plane.

---

*Sketch of Proof*: Begin with |z|<R. Then

$$\left|f_n(z)\right|\le\alpha_n\left|z\right|\left[\frac{\alpha_n-1}{\alpha_n}+\frac{1}{\alpha_n}\left(1+\rho_n\left|z\right|+\rho_n^2\left|z\right|^2+\cdots\right)\right]$$

$$\le \left|z\right|\left[\left|\alpha_n-1\right|+\frac{1}{1-\rho_n\left|z\right|}\right]\le\left|z\right|\left[\varepsilon_n+\frac{1}{1-\rho_n\left|z\right|}\right]$$

$$\le \frac{\left(1+\varepsilon_n\right)\left|z\right|}{1-\rho_n\left|z\right|}$$

Thus,

$$(1)\quad \left|f_n(z)\right| \le \frac{(1+\varepsilon_n)|z|}{1-\rho_n|z|}$$

Repeated application of (1) gives

$$\left|f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z)\right| \le \frac{R\prod_n^{n+m}(1+\varepsilon_k)}{1-R\left[\rho_{n+m}+\rho_{n+m-1}(1+\varepsilon_{n+m})+\cdots+\rho_n\prod_n^{n+m}(1+\varepsilon_k)\right]}$$

Where $\prod_n^{n+m}(1+\varepsilon_k) < M$ and n sufficiently large $\Rightarrow \sum_n^{\infty}\rho_k < \frac{1}{2MR}$

Hence $\left|f_n \circ \cdots \circ f_{n+m}(z)\right| \le \dfrac{MR}{1-MR\sum_n^{n+m}\rho_k} < R_0 \equiv 2MR$ .

It is easily seen that

$$(2)\quad \left|f_n(z)-z\right| \le \varepsilon_n|z| + \rho_n \cdot \frac{|z|^2}{1-\rho_n|z|}.$$

And, finally,

$$\begin{aligned}
\left|f_n(z)-f_n(\zeta)\right| &\le |z-\zeta|\cdot\left[\alpha_n+\rho_n(|z|+|\zeta|)+\rho_n^2(|z|^2+|z||\zeta|+|\zeta|^2)+\cdots\right] \\
&\le |z-\zeta|\cdot\left[\varepsilon_n+\left(1+2\rho_n R_0+3\rho_n^2 R_0{}^2+\cdots\right)\right] \\
&\le |z-\zeta|\cdot\left[\varepsilon_n+\frac{1}{(1-\rho_n R_0)^2}\right]
\end{aligned}$$

Leading to

$$(3)\quad \left|f_n(z)-f_n(\zeta)\right| \le (1+\beta_n)\cdot|z-\zeta| \text{ , where } \sum\beta_n < \infty$$

As before, using (1), (2), (3) and setting $F_{n,n+m} = f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z)$

$$\left|F_{n,n+m}-F_{n,n+m+p}\right| \le \prod_{k=n}^{n+m}(1+\beta_k)\cdot\left|F_{n+m+1,n+m+p}-z\right|$$

$$\le M\Big\{\left|f_{n+m+1}(F_{n+m+2,n+m+p})-F_{n+m+2,n+m+p}\right|$$

$$+\left|f_{n+m+2}(F_{n+m+3,n+m+p})-F_{n+m+3,n+m+p}\right|$$

$$+\cdots+\left|f_{n+m+p}(z)-z\right|\Big\}$$

Therefore, for sufficiently large and fixed n:

$$\left|F_{n,n+m}-F_{n,n+m+p}\right| \le M\cdot\left[R_0\sum_{n+m+1}^{n+m+p}\varepsilon_k+\frac{R_0^2}{1-\frac{1}{2}}\sum_{n+m+1}^{n+m+p}\rho_k\right] \to 0 \text{ as } m\to\infty \quad ||$$

Another simple result:

---

Theorem 2.6 (*Gill* 2011) Consider functions $\{f_n(z)\}$ analytic for $|z|\le R_0$. If $|f_n(z)-z|\le C\rho^n$ for $0\le\rho<1$ and $|z|\le R_0=R+\sigma$, where $\sigma=C\dfrac{\rho}{1-\rho}$, then $\lim\limits_{n\to\infty}\left(f_1\circ f_2\circ\cdots\circ f_n(z)\right)=F(z)$ for $|z|\le R$. Convergence is uniform.

---

*Sketch of Proof*: It is easily seen that

$$\left|f_n\circ f_{n+1}\circ\cdots\circ f_{n+m}(z)\right|\le R+\sigma=R_0, \text{ starting with } |z|\le R$$

Now, $f_n(z)\to f(z)\equiv z$ uniformly on $(|z|\le R_0)=S$, so that $f_n'(z)\to f'(z)\equiv 1$ uniformly on $S$. Hence for large values of n, $|f_n'(z)|\le 1+\delta_n\approx 1$.
Therefore, from the fundamental theorem of calculus,
$|f_n(z)-f_n(\zeta)|\le(1+\delta_n)|z-\zeta|$ for $z,\zeta\in S$. Now, choose and fix a value of n large enough to insure $\rho<\dfrac{1}{1+\delta_n}$. Then, for z in $S'=(|z|\le R)$,

$$\left|f_n\circ\cdots\circ f_{n+m}(z)-f_n\circ\cdots\circ f_{n+m+p}(z)\right|\le(1+\delta_n)^m C\left[\rho^{n+m+1}+\rho^{n+m+2}+\cdots\right]$$

$$\le\frac{C\rho^{n+1}}{1-\rho}\left[\rho(1+\delta_n)\right]^m\to 0 \text{ as } m\to\infty$$

Hence $f_1\circ\cdots\circ f_{n-1}(f_n\circ\cdots\circ f_{n+m}(z))\to F(z)$ as $m\to\infty$, uniformly on $(|z|\le R)$ ||

*Comments*: That functions more sophisticated than $f_n(z) = z + C\rho^n$ are possible, we have

*Example*: $f_n(z) = z + \rho^n \dfrac{1}{Q+z}$ with $Q > 2$. Set $R = \dfrac{Q}{2}$ .

Let $F_{n,n+m}(z) = f_n \circ f_{n+1} \circ \cdots \circ f_{n+m}(z)$ ; then backward recursion gives

$$\left|F_{n,n+m}(z)\right| \le \frac{Q}{2} + C\frac{\rho}{1-\rho} = R_1$$

Thus $\left|f_n(z) - z\right| \le \rho^n \dfrac{1}{Q - R_1} \le \rho^n$ with $\rho < 1 - \dfrac{2}{Q}$.

Expanding upon Theorem 2.2 (actually, can be seen as a corollary to Theorem 2.6):

---

Theorem 2.7 (Gill, 2011) Suppose $f_n(z) = z\left(1+\eta_n(z)\right)$, with $\eta_n$ analytic for $|z| \le R_1$ and

$|\eta_n(z)| < \varepsilon_n$ , $\sum \varepsilon_n < \infty$ . Choose $0 < r < R_1$, and define $R = R(r) = \dfrac{R_1 - r}{\prod_{k=1}^{\infty}\left(1+\varepsilon_k\right)}$

Then $F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z) \rightarrow F(z)$ uniformly for $|z| \le R$

and $|F'(z)| \le \prod_{k=1}^{\infty}\left(1+\beta_k\right) < \infty$ where $\beta_k = \dfrac{R_1}{r}\varepsilon_k$ .

---

*Proof*: Simple recursion shows that

$$\left|f_n \circ \cdots \circ f_{n+m}(z)\right| \le R \cdot \prod_{k=n}^{n+m}\left(1+\varepsilon_k\right) \le R \cdot \prod_{k=1}^{\infty}\left(1+\varepsilon_k\right) = R^*.$$

Next, a Lipshitz condition is required: But first, Cauchy's Integral Formula provides an estimate of the derivatives: $|\eta_n{}'(z)| \le \dfrac{\varepsilon_n}{r}$. Write

$$|f_n(z) - f_n(\zeta)| \le |z - \zeta| + |z\eta_n(z) - \zeta\eta_n(\zeta)| \quad \text{where}$$

$$|z\eta_n(z) - \zeta\eta_n(\zeta)| = \left|\int_\zeta^z d(t\eta_n(t))\right| = \left|\int_\zeta^z \left(\eta_n(t) + t\eta_n'(t)\right)dt\right|$$

$$\le \int_\zeta^z \left(|\eta_n(t)| + |t|\cdot\left|\eta_n'(t)\right|\right)|dt| \le \left(\varepsilon_n + (R_1 - r)\frac{\varepsilon_n}{r}\right)\cdot|z-\zeta| = \frac{R_1}{r}\varepsilon_n|z-\zeta| = \beta_n \cdot |z-\zeta|$$

Then $|F_n(z) - F_n(\zeta)| \le \prod_{k=1}^{n}\left(1+\beta_k\right)\cdot|z-\zeta|$ , with the product converging.

Also $\left|f_n(z)-z\right| \le |z|\cdot\left|\eta_n(z)\right| \le R^*\varepsilon_n$. Therefore

$$\left|F_n(z)-F_{n+m}(z)\right| \le \prod_{k=1}^{n}\left(1+\beta_k\right)\cdot\left|f_{n+1}\circ\cdots\circ f_{n+m}(z)-z\right|$$

Set $Z_{n+1}^{n+m} = f_{n+1}\circ\cdots\circ f_{n+m}(z)$. Then

$$\begin{aligned}\left|f_{n+1}\circ\cdots\circ f_{n+m}(z)-z\right| &\le \left|f_{n+1}(Z_{n+2}^{n+m})-Z_{n+2}^{n+m}\right| + \left|f_{n+2}(Z_{n+3}^{n+m})-Z_{n+3}^{n+m}\right| + \cdots + \left|f_{n+m}(z)-z\right| \\ &\le R^*\varepsilon_n + R^*\varepsilon_{n+1} + \cdots + R^*\varepsilon_{n+m} \le R^*\sum_{k=n}^{n+m}\varepsilon_k \;\to\; 0 \quad \text{as n} \to \infty\end{aligned}$$

Hence

$$\left|F_n(z)-F_{n+m}(z)\right| \le R^*\prod_{k=1}^{\infty}\left(1+\beta_k\right)\cdot\sum_{k=n}^{\infty}\varepsilon_k \;\to\; 0 \text{ uniformly on } \left(|z|\le R\right) \text{ as n} \to \infty$$

A simple application of the chain rule gives $\left|F'(z)\right| \le \prod_{k=1}^{\infty}\left(1+\beta_k\right) < \infty$

*Example & Comments*: The following example shows how the hypothesis can be realized.

Set $\quad f_n(z) = z\left(1+\dfrac{1}{C(n+1)^2+z}\right)$, $C>1$, $|z|\le C$. Then

$$\left|f_n(z)\right| \le C\left(1+\frac{1}{C\left((n+1)^2-1\right)}\right) \le C\left(1+\frac{1}{Cn^2}\right) \quad , \quad \text{so that}$$

$$\begin{aligned}\left|f_{n+m-1}\circ f_{n+m}(z)\right| &\le C\left(1+\frac{1}{C(n+m)^2}\right)\cdot\left(1+\frac{1}{C\left[(n+m)^2-(1+\frac{1}{C(n+m)^2})\right]}\right) \\ &\le C\left(1+\frac{1}{C(n+m-1)^2}\right)\left(1+\frac{1}{C(n+m)^2}\right), \text{ etc.}\end{aligned}$$

Thus $\varepsilon_n = \dfrac{1}{Cn^2}$ and $\quad R^* = C\displaystyle\prod_{k=1}^{\infty}\left(1+\frac{1}{Ck^2}\right)$

**Outer Compositions**

---

Theorem 2.8 (Gill, 2010) Consider functions $g_n(z) = a_n z + c_{n,2}z^2 + c_{n,3}z^3 + \cdots$ with complex coefficients.

$$\text{Set} \quad \rho_n = \sup_{r=2,3,4\ldots} \left\{ \left| c_{n,r} \right|^{\frac{1}{r-1}} \right\} \quad \text{and} \quad \varepsilon_n = \left| a_n - 1 \right| \quad \text{with} \sum \varepsilon_n < \infty$$

If $\rho_n < \dfrac{\delta_n}{RM_1M_2}$ , where $\sum \delta_n < \infty$ , $\prod (1+\delta_n) < M_1$ , $\prod (1+\varepsilon_n) < M_2$ ,

then

$$G(z) = \lim_{n\to\infty} \left( g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \right) \text{ exists and is analytic for } |z|<R .$$

Convergence is uniform on compact subsets of ($|z|<R$).

---

*Sketch of Proof*:

$$\left| g_n(z) \right| \le \left| (a_n - 1) z \right| + |z| + \rho_n |z|^2 + \rho_n^2 |z|^3 + \cdots$$

$$\le \varepsilon_n |z| + \frac{|z|}{1 - \rho_n |z|} \quad \le \quad \frac{(1+\varepsilon_n)|z|}{1-\rho_n |z|}$$

Repeated application gives

$$\left| g_{n+m} \circ g_{n+m-1} \circ \cdots \circ g_n(z) \right| \;\le\; R \cdot \prod_n^{n+m} (1+\delta_k) \cdot \prod_n^{n+m} (1+\varepsilon_k) \;<\; R_0$$

Fix n so large that $\displaystyle\sum_n^{\infty} \rho_k < \frac{1}{2R_0}$.

Next $$\left| g_n(z) - z \right| \;\le\; \varepsilon_n |z| \;+\; \rho_n \frac{|z|^2}{1-\rho_n |z|}$$

Then, setting $G_n(z) = G_n = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$

$$\left|G_{n+m}-G_n\right| \le \left|g_{n+m}(G_{n+m-1})-G_{n+m-1}\right| + \left|g_{n+m-1}(G_{n+m-2})-G_{n+m-2}\right|$$

$$+ \quad \cdots \quad + \quad \left|g_{n+1}(G_n)-G_n\right|$$

$$\le R_0\varepsilon_{n+m} + \rho_{n+m}\frac{R_0^2}{1-\frac{1}{2}} + \cdots + R_0\varepsilon_{n+1} + \rho_{n+1}\frac{R_0^2}{1-\frac{1}{2}}$$

$$\le R_0\sum_{n+1}^{\infty}\varepsilon_k + 2R_0^2\sum_{n+1}^{\infty}\rho_k \quad \to \quad 0 \text{ as } n \to \infty$$

In a more general context:

---

Theorem 2.9 (Gill, 2011) Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty}\rho_n < \infty$ and $\left|g_n(z)-z\right| < C\rho_n$ if $|z|<M$. Set $\sigma = C\sum_{1}^{\infty}\rho_k$ and $R_0 = M-\sigma > 0$. Then, for every $z \in S_0 = \left(|z|<R_0\right)$, $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \ \to\ G(z)$, uniformly on compact subsets of $S_0$.

---

*Comment*: There is no requirement these functions be analytic.

*Sketch of Proof*: $\left|G_1(z)\right| < |z| + C\rho_1 < R_0 + C\rho_1$ for $|z| < R_0$. Repeated applications give

$$\left|G_n(z)\right| < R_0 + C(\rho_1+\cdots+\rho_n) < R_0 + \sigma = M. \qquad \text{Thus}$$

$$\left|G_{n+m}-G_n\right| \le \left|g_{n+m}(G_{n+m-1})-G_{n+m-1}\right| + \left|g_{n+m-1}(G_{n+m-2})-G_{n+m-2}\right| + \cdots + \left|g_{n+1}(G_n)-G_n\right|$$

$$< \ C\sum_{k=n+1}^{n+m}\rho_k \ \le \ C\sum_{k=n+1}^{\infty}\rho_k \ \to\ 0 \text{ as } n \to \infty$$

*Example*: $\qquad g_n(z) = z + \rho_n \dfrac{z}{5+z} \quad$ for $|z| \le 3$.

We have $|g_n(z)| \le |z| + \rho_n \dfrac{|z|}{5-|z|} < 3 + 2\rho_n$ , and repeated applications give

$|g_{n+m} \circ \cdots \circ g_n(z)| < 3 + 2\sum_{k=n}^{\infty} \rho_k < 4$ for $\sum_{k=1}^{\infty} \rho_k < \dfrac{1}{6}$. Thus $|g_n(z) - z| \le 4\rho_n$ , and

$|G_{n+m} - G_n| \le |g_{n+m}(G_{n+m-1}) - G_{n+m-1}| + \cdots + |g_{n+1}(G_n) - G_n| \le 4\sum_{k=n+1}^{\infty} \rho_k \to 0$ as $n \to \infty$

I.e., $z = 2$ and $\rho_n = \left(\dfrac{1}{8}\right)^n$ gives G(14)=2.040882... error<$10^{-14}$.

*Example*: $g_n(z) = z + \rho^n \int_0^1 \dfrac{e^{tz}}{t+1} dt$ for $|z| \le 1$ with $\rho < \dfrac{2}{e}$. We have

$|g_n(z)| \le |z| + \rho^n \int_0^1 \left| \dfrac{e^{tz}}{t+1} \right| dt \le |z| + \rho^n \dfrac{e^{|z|}}{2} \le 1 + \rho^n \dfrac{e}{2}$ so that

$|g_{n+m} \circ g_{n+m-1} \circ \cdots \circ g_n(z)| \le 1 + \dfrac{e}{2}\rho^n + \dfrac{e^2}{2}\rho^{n+1} + \cdots \le M_n$. Thus

$|g_n(z) - z| \le \rho^n \cdot \dfrac{e^{M_1}}{2} = C\rho^n$. Hence

$$
\begin{aligned}
|G_{n+m} - G_n| &\le |g_{n+m}(G_{n+m-1}) - G_{n+m-1}| + \cdots + |g_{n+1}(G_n) - G_n| \\
&\le C\left[\rho^{n+1} + \cdots + \rho^{n+m}\right] \to 0 \text{ as } n \to \infty
\end{aligned}
$$

**Generated self-replicating series**

*Example*: Consider a wandering point in the complex plane whose position depends upon its previous position. For example:

$g_n(z) = z + \dfrac{1}{\rho n^2}\sqrt{z}$ , $\rho > \sqrt{\dfrac{\pi}{6}}$ and $|z| < R = \rho^2 - \dfrac{\pi}{6}$ ,$\mathrm{Re}(z) > 0$. Then

$G_n(z) = z + a_1(z) + a_2(z) + \cdots + a_n(z)$ with $a_n(z) = \dfrac{1}{\rho n^2}\sqrt{G_{n-1}(z)}$

And Theorem 2.9 applies, giving $\lim_{n\to\infty} G_n(z) = G(z)$.

Theorem 2.10 (Gill, 2011) Suppose $g_n(z)=z\left(1+\eta_n(z)\right)$, with $\eta_n$ analytic for $|z|\le R_1$ and $|\eta_n(z)|<\varepsilon_n$ , $\sum\varepsilon_n<\infty$ . Choose $0<r<R_1$, and define $R=R(r)=\dfrac{R_1-r}{\prod_{k=1}^{\infty}\left(1+\varepsilon_k\right)}$ .

Then $G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)\to G(z)$ uniformly for $|z|\le R$

with $|G'(z)|\le\prod_{k=1}^{\infty}\left(1+\dfrac{R_1}{r}\varepsilon_k\right)$ and $|G'(0)|\le\prod_{k=1}^{\infty}\left(1+\varepsilon_k\right)$.

---

*Sketch of Proof*: It is easily seen that $|G_n(z)|\le R\cdot\prod_{k=1}^{n}\left(1+\varepsilon_k\right)\le M$

$$|G_{n+1}(z)-G_n(z)|=|g_{n+1}(G_n(z))-G_n(z)|\le M\varepsilon_{n+1}$$
,
$$|G_{n+m}(z)-G_n(z)|\le M\cdot\sum_{k=n+1}^{n+m}\varepsilon_k\to 0 \text{ as n}\to\infty$$

In the proof of Theorem 2.7 it is shown that $|g_n(z)-g_n(\zeta)|\le\left(1+\beta_n\right)\cdot|z-\zeta|$

Hence $$\left|\frac{G_n(z+t)-G_n(t)}{t}\right|\le\prod_{k=1}^{n}\left(1+\beta_k\right)\text{ , and}$$

$$\left|\frac{G_n(z)-G_n(0)}{z}\right|=\left|\frac{G_n(z)}{z}\right|=\left|\frac{g_n(G_{n-1})}{G_{n-1}}\right|\cdot\left|\frac{g_{n-1}(G_{n-2})}{G_{n-2}}\right|\cdots\left|\frac{g_1(z)}{z}\right|\le\prod_{k=1}^{n}\left(1+\varepsilon_k\right)\quad\|$$

*Example*: $g_n(z)=z\left(1+\dfrac{1}{C(n+1)^2+z}\right)$, $C>1$, $|z|\le C$. Then

$$|g_n(z)|\le C\left(1+\frac{1}{C\left((n+1)^2-1\right)}\right)\le C\left(1+\frac{1}{Cn^2}\right)\ ,$$

$$|g_{n+1}\circ g_n(z)|\le C\left(1+\frac{1}{Cn^2}\right)\cdot\left(1+\frac{1}{C\left[(n+2)^2-(1+\frac{1}{Cn^2})\right]}\right)\le C\left(1+\frac{1}{Cn^2}\right)\left(1+\frac{1}{C(n+1)^2}\right),$$

.
.
.

$|G_{n,n+m}(z)|\le C\cdot\prod_{k=n}^{n+m}\left(1+\dfrac{1}{Ck^2}\right)\le M$ . Hence $G_n(z)\to G(z)$.

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