Expanding Functions into Infinite Compositions

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(Preliminary Discussion)

Infinite compositions of analytic functions may occur in two forms:

I Inner or right compositions: $\mathop{\mathcal{R}}\limits_{k=1}^{n} t_k(z) = t_1 \circ t_2 \circ \cdots \circ t_n(z)$, $T(z) = \lim\limits_{n\to\infty} \mathop{\mathcal{R}}\limits_{k=1}^{n} t_k(z)$.

II Outer or left compositions: $\mathop{\mathcal{L}}\limits_{k=1}^{n} t_k(z) = t_n \circ t_{n-1} \circ \cdots \circ t_1(z)$, $T(z) = \lim\limits_{n\to\infty} \mathop{\mathcal{L}}\limits_{k=1}^{n} t_k(z)$.

Convergence theory of each of these may be found in [1] and [2]. Here, the emphasis will be on finding algorithms for converting closed form expressions into infinite expansions. Simple functional equations that relate a function of nz to an expression of the same function of z sometimes lead directly to such expansions. Consider the example:

Example Ia $Tan(2z) = \dfrac{2Tan(z)}{1-Tan^2(z)}$. We follow the procedure (Kojima [2]):

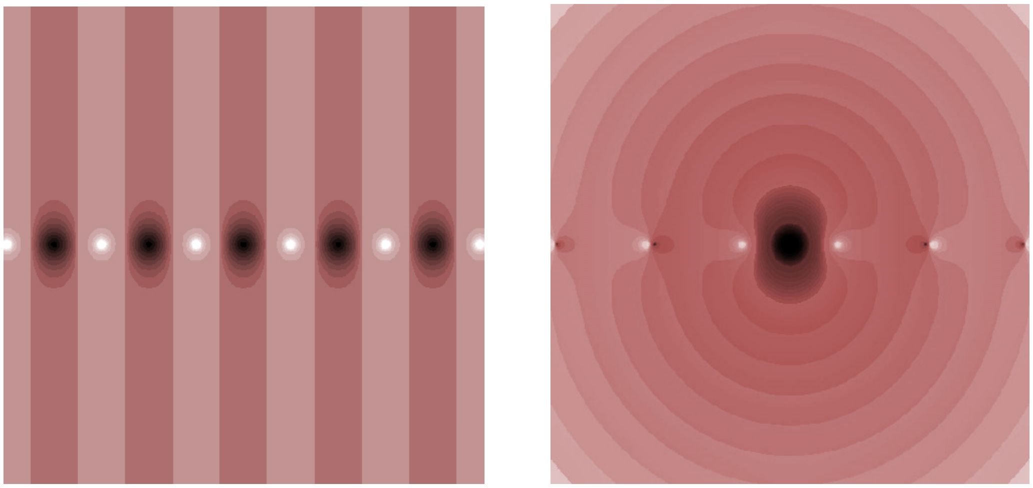
$$T(z) = Tan(z) = 2\frac{z}{1-z^2} \circ Tan(\tfrac{z}{2}) = \frac{z}{1-\frac{1}{4}z^2} \circ 2z \circ Tan(\tfrac{z}{2}) = \frac{z}{1-\frac{1}{4}z^2} \circ 4\frac{z}{1-z^2} \circ Tan(\tfrac{z}{4})$$
$$= \frac{z}{1-\frac{1}{4}z^2} \circ \frac{z}{1-\frac{1}{4^2}z^2} \circ 4z \circ Tan(\tfrac{z}{4}) = \cdots$$

Which leads to $T(z) = \mathop{\mathcal{R}}\limits_{k=1}^{n} t_k(z) = t_1 \circ t_2 \circ \cdots \circ t_n(r_n(z))$ with

$$t_k(z) = \frac{z}{1-\frac{1}{4^k}z^2} \quad , \quad r_n(z) = 2^n Tan(z/2^n) \to z$$

Computer experiments lead to the conjecture $Tan(z) = \lim\limits_{n\to\infty} \mathop{\mathcal{R}}\limits_{k=1}^{n} t_k(z) = \mathop{\mathcal{R}}\limits_{k=1}^{\infty} t_k(z)$.

Image Ia Tan(z) (n=5) and Tan(z) – z (n=20) [-8<x,y<8]



Convergence in a neighborhood of $z=0$ can be seen by applying the following

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Theorem 2.6 [1] Suppose $f_n(z)=z\left(1+\eta_n(z)\right)$, with η_n analytic for $|z|\le R_1$ and

$|\eta_n(z)|<\varepsilon_n$, $\sum \varepsilon_n<\infty$. Choose $0<r<R_1$, and define $R=R(r)=\dfrac{R_1-r}{\prod_{k=1}^{\infty}(1+\varepsilon_k)}$ Then

$F_n(\mathrm{z})=f_1\circ \mathrm{f}_2\circ\cdots\circ \mathrm{f}_n(\mathrm{z}) \ \to\ \mathrm{F(z)}$ uniformly for $|z|\le R$ and $|F'(z)|\le \prod_{k=1}^{\infty}(1+\beta_k)<\infty$ where

$\beta_k=\dfrac{R_1}{r}\varepsilon_k$.

Here $|\eta_n(z)|=\dfrac{1}{4^n}\dfrac{|z|^2}{\left|1-\frac{1}{4^n}z^2\right|}$. The actual region of convergence is larger.

Although there are many examples showing the convergence of an inner or right composition to the function of which it is an expansion (see, e.g., analytic theory of continued fractions [3]), there are perhaps no previous non-trivial examples showing the same for outer or left compositions.

Here is one:

Example IIa Set $g_k(z) = t_k^{-1}(z)$, $\gamma_n(z) = r_n^{-1}(z)$ (Example Ia). Then $g_k(z) \to z$, $\gamma_n(z) \to z$ and

$$Arc\tan(z) = \gamma_n \circ g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \approx g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$$

with $g_k(z) = \dfrac{2 \cdot 4^k}{z}\left(\sqrt{1+\frac{1}{4^k}z^2} - 1\right)$. Set $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$.

Hence, the conjecture $G_n(z) \to Arc\tan(z)$, or

$$Arc\tan(z) = \mathop{\mathcal{L}}_{k=1}^{\infty} \frac{2 \cdot 4^k}{z}\left(\sqrt{1+\tfrac{1}{4^k}z^2} - 1\right),$$ P.V. of course, which is supported by computer examples. This may be written in a simpler form:

$$Arc\tan(z) = \mathop{\mathcal{L}}_{k=0}^{\infty}\left(\frac{2z}{1+\sqrt{1+\zeta_k z}}\right) \quad , \quad \zeta_k = \frac{z}{4^k} \quad .$$

Convergence in a neighborhood of $z = 0$ can be verified by employing the following

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Theorem 2.8 [1] Let $\{g_n\}$ be a sequence of complex functions defined on $S=(|z|<M)$. Suppose there exists a sequence $\{\rho_n\}$ such that $\sum_{k=1}^{\infty} \rho_n < \infty$ and $|g_n(z) - z| < C\rho_n$ if $|z| < M$. Set $\sigma = C\sum_1^{\infty} \rho_k$ and $R_0 = M - \sigma > 0$. Then, for every $z \in S_0 = (|z| < R_0)$, $G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z) \to G(z)$, uniformly on compact subsets of S_0 .

Here $|g_n(z) - z| = \dfrac{1}{4^n}\dfrac{|z|^3}{\left|1+\sqrt{1+\frac{1}{4^n}z^2}\right|}$. The actual convergence region is larger.

Example Ib $F(z) = e^z - 1$. $F(2z) = F(z)(F(z)+2)$ gives

$$F(z) = z(z+2) \circ F(z/2) = \left(\frac{z^2}{4} + z\right) \circ 2z \circ F(z/2) = \left(\frac{z^2}{4} + z\right) \circ 2F(z/2)$$

$$= \left(\frac{z^2}{4} + z\right) \circ \left(2z^2 + 4z\right) \circ F(z/4) = \left(\frac{z^2}{4} + z\right) \circ \left(\frac{z^2}{8} + z\right) \circ 4z \circ F(z/4) = \cdots$$

With $r_n(z) = 2^n F\left(\frac{z}{2^n}\right) \to z$. We have the following

$$e^z = 1 + \underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(\frac{z^2}{2^{k+1}} + z\right)$$

Theorem 2.6[1] can be used to show convergence in a neighborhood of $z = 0$ with $\eta_n(z) = \frac{z}{2^{n+1}}$.

Of course, the radius of convergence of the composition is actually infinite.

Example IIb In the previous example $t_k(z) = \frac{z^2}{2^{k+1}} + z$, so that

$g_k(z) = t_k^{-1}(z) = \frac{2z}{1+\sqrt{1+4\left(\frac{1}{2^{k+1}}\right)z}}$. Thus $Ln(z+1) \approx G_n(z) = g_n \circ g_{n-1} \circ \cdots \circ g_1(z)$ and

$Ln(z+1) = \underset{k=1}{\overset{\infty}{\mathcal{L}}}\left(\frac{2z}{1+\sqrt{1+4\left(\frac{1}{2^{k+1}}\right)z}}\right)$. $n = 50$ gives ten decimal place accuracy for $Ln(6-8i)$.

Image 2a Ln(1+z) - z , n=10, -8<x,y<8 and Ln(1+z)

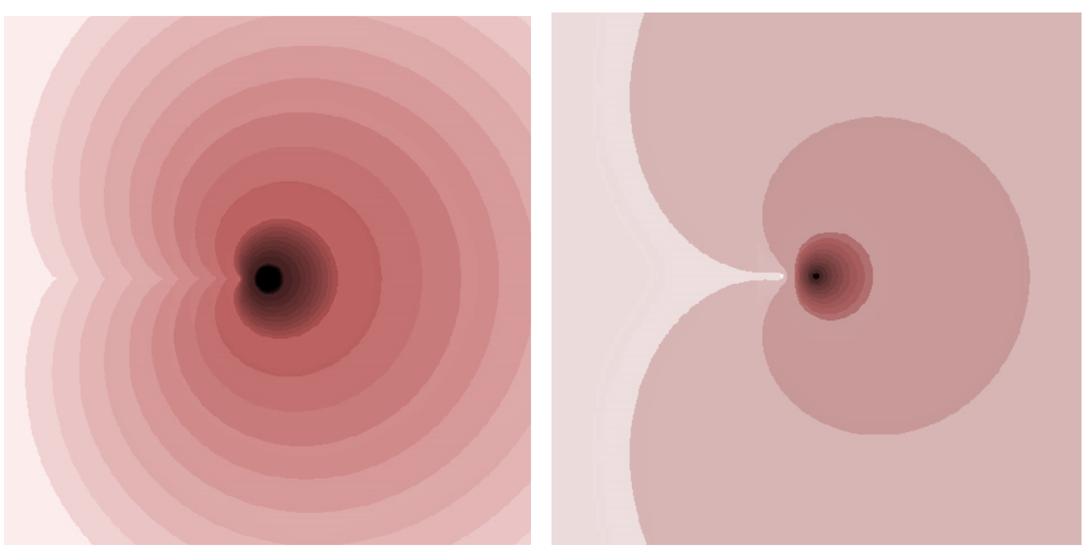
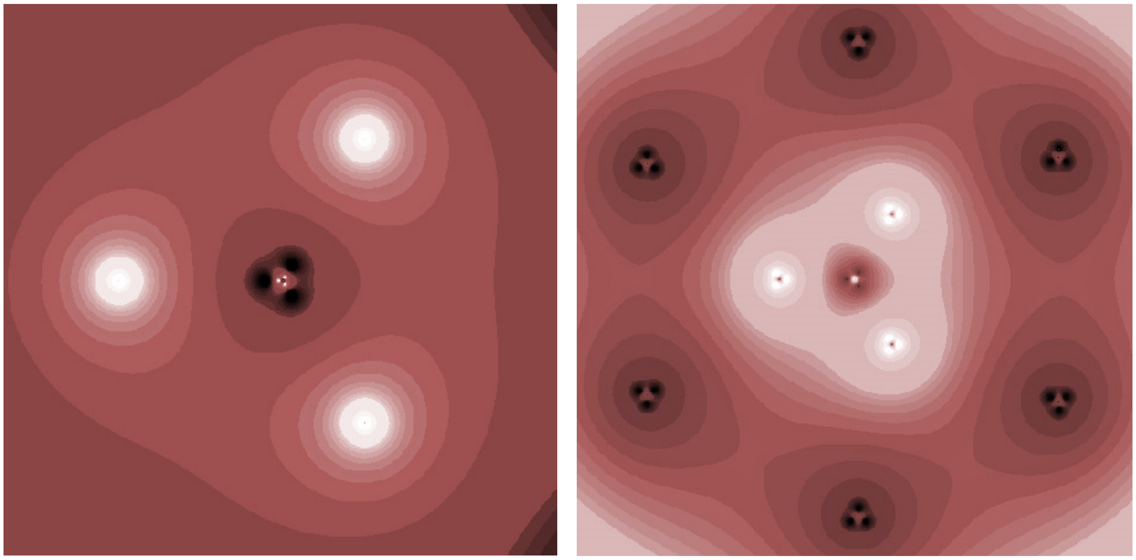


Image 2b $F(z) = \underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(\frac{z^2}{10^k}+\frac{1}{z}\right)$ n=5 -8<x,y<8 $G_{10}(z) = \underset{k=0}{\overset{10}{\mathcal{L}}}\left(\frac{z^2}{10^k}+\frac{1}{z}\right)$ non-convergent



Example Ic $F(z) = Sin(z)$

$$Sin(z) = 2z\sqrt{1-z^2}\circ Sin(z/2) = z\sqrt{1-\frac{1}{4}z^2}\circ 2Sin(z/2) = z\sqrt{1-\frac{1}{4}z^2}\circ 4z\sqrt{1-z^2}\circ Sin(z/4)$$

$$= z\sqrt{1-\frac{1}{4}z^2}\circ z\sqrt{1-\frac{1}{4^2}z^2}\circ 4Sin(z/4) = \cdots$$

with $r_n(z) = 2^n Sin(z/2^n) \to z$. We have

$$Sin(z) = \pm\underset{k=1}{\overset{\infty}{\mathcal{R}}}\left(z\sqrt{1-\frac{1}{4^k}z^2}\right)$$

Where the positive sign is valid in Q1 and Q4 and the negative sign in Q2 and Q3. For $Sin(1+4i)$ the value is accurate to ten decimal places for $n = 20$.

Continued Fractions (CFs) are a special case of inner composition (I), involving two complex variables.

One type is $F(z) = \cfrac{a_1(z)}{1+\cfrac{a_2(z)}{1+\cfrac{a_3(z)}{1+\ddots}}}$, defined by $t_n(z;\zeta) = \frac{a_n(z)}{1+\zeta}$ and

$T_1(z;\zeta) = t_1(z;\zeta)$, $T_n(z;\zeta) = T_{n-1}(z;t_n(z;\zeta))$. Then

$$F(z)=\lim_{n\to\infty}T_n(z;\zeta) \text{ or } \mathcal{R}_{n=1}^{\infty}\left(t_n(z;\zeta)\right)_{\zeta=0}.$$

Although $\zeta=0$ normally, other values of the variable ζ frequently lead to the same value of $F(z)$. The essential difference between the examples cited previously and CFs is that the former represent compositions on z that lead to functions $F(z)$, whereas the latter evolve from compositions on an "auxiliary" variable ζ, leading to $F(z)$.

Example Id $t_n(z;\zeta)=\dfrac{a_n(z)}{1+\zeta}$ where $|\zeta|<R$ $\left(R\le\dfrac{1}{2}\right)$ and $|a_n(z)|<\rho R(1-R)$, $0<\rho<1$, with $a_n(z)$ analytic for $z\in S$. Then $|t_n(z;\zeta)|<\rho R$ and these functions contract uniformly.

Therefore $F(z)=\mathcal{R}_{n=1}^{\infty}\left(\dfrac{a_n(z)}{1+\zeta}\right)_{\zeta}$, analytic for $|\zeta|<R$ and $z\in S$ (see Contraction Theorems in [1]).

III Implicit functions and Zeno contours: Consider an expression defining a function implicitly:

$$\Phi\left(\zeta,f(\zeta)\right)=0 \quad \text{or} \quad \Phi\left(\zeta,z\right)=0\ ,\ z=f(\zeta).$$

The following definition is from [4]:

Zeno contour: Let $g_{k,n}(z)=z+\eta_{k,n}\varphi(z)$ where $z\in S$ and $g_{k,n}(z)\in S$ for a convex set S in the complex plane. Require $\lim_{n\to\infty}\eta_{k,n}=0$, where (usually) $k=1,2,...,n$. Set $G_{1,n}(z)=g_{1,n}(z)$, $G_{k,n}(z)=g_{k,n}\left(G_{k-1,n}(z)\right)$ and $G_n(z)=G_{n,n}(z)$ with $G(z)=\lim_{n\to\infty}G_n(z)$, when that limit exists.

The *Zeno contour* is a graph of this iteration. Normally, for a vector field, $\mathbb{F}=F$, $\varphi(z)=F(z)-z$, and under the right conditions $G(z)=\alpha$, an attractive fixed point of F.

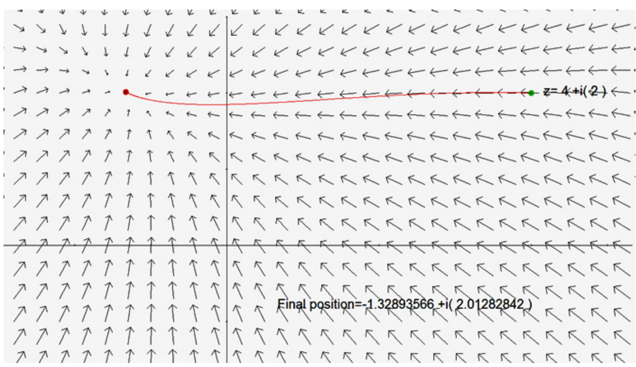
In the context of this discussion $g_{k,n}(z)=z+\eta_{k,n}\left(F\left(\zeta,z\right)-z\right)$, and if $\left|\dfrac{\partial F}{\partial z}\right|<\rho<1\ \ \forall\zeta\in S$, a suitable domain, then $G_n(z)\to\alpha(\zeta)=f(\zeta)$ a fixed point for each value of ζ, starting with an initial value z in some neighborhood of the fixed points [4]. Thus, from the notation II,

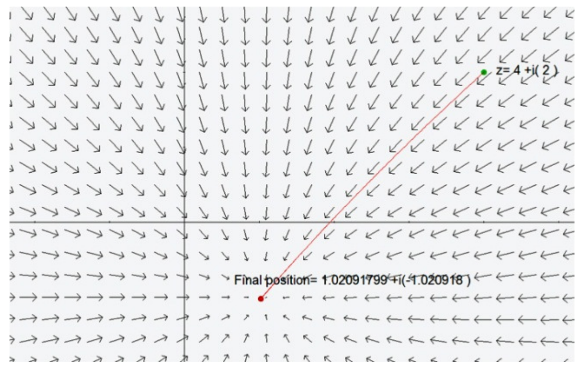
III $$\mathcal{L}_{k=1}^{n}g_{k,n}(z)=g_{n,n}\circ g_{n-1,n}\circ\cdots\circ g_{1,n}(z) \quad \text{and} \quad G(z)=\lim_{n\to\infty}\mathcal{L}_{k=1}^{n}g_{k,n}(z).$$

Example IIIa: $\Phi(\zeta,z)=\zeta Cos\left(\frac{\zeta z}{10}\right)+z=0$. Then $F(\zeta,z)=-\zeta Cos\left(\frac{\zeta z}{10}\right)$ and the Zeno contour terminates at $z=f(\zeta)$ for ζ in a neighborhood of the origin and initial values of z near the fixed points. For example,

$f(1-2i)\approx -1.3289+i(2.0128)$ and $f(-1+i)\approx 1.0209+i(-1.0209)$,

starting with $z_0=4+2i$.





References

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[2] S. Kojima, Convergence of infinite compositions of entire functions, arXiv:1009.2833v1

[3] L. Lorentzen, H. Waadeland, *Continued Fractions with Applications*, North Holland (1992)

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