A Fixed-Point Algorithm for Functions defined by Infinite Expansions

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A general example illustrates the algorithm (*Gill*, [1] 1991):

The analytic continued fraction

(1) $$G(z)=\frac{a_1(z)}{b_1(z)+}\ \cdots\ \frac{a_n(z)}{+b_n(z)+}\ \cdots$$

converges under a variety of conditions, including the following: Set

$$t_n(z,\zeta)=\frac{a_n(z)}{b_n(z)+\zeta}\ ,\ T_1(z,\zeta)=t_1(z,\zeta)\ ,\ T_n(z,\zeta)=T_{n-1}(z,t_n(z,\zeta))$$

Let $t_n(z,\zeta)$ be defined as an analytic function (in each variable) on $S\times S$, for S a simply-connected domain containing 0. Suppose there exists a compact set $\Omega\subset S$ where $t_n(S,S)\subset\Omega$. Then each such function is a *contraction mapping* in each variable. The *nth convergent* of (1) may be written as

(2) $$g_n(z)=T_n(z,\zeta_0)=t_1\circ t_2\circ\cdots\circ t_n(\zeta_0)$$, a *forward iteration* or *inner composition*,

where the composition is on the second variable. Normally $\zeta_0=0$ if 0 is in S, as would be the case for instance if S = (|z|<R). Now set

$$G_n(z)=g_n\circ g_{n-1}\circ\cdots\circ g_1(z)$$, a *backward iteration* or *outer composition*.

Here the composition is on the variable z.

It is shown in (*Gill*, [1], 1991) that

$$G_n(z)\ \to\ \alpha=G(\alpha)$$, the *attractive fixed point* of (1) in S.

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In the following, details have been omitted:

Example 1 : let $G(z) = \frac{\frac{e^z}{4}}{3+z+} \frac{\frac{e^z}{8}}{3+z+} \frac{\frac{e^z}{12}}{3+z+} \cdots$, where $|z| \le 1$. Applying the algorithm, starting with $z = 1$, one obtains $\alpha = .087118118\cdots$ to ten decimal places after ten iterations.

Example 2 : Solve the equation $\mathrm{Tan}(z^2) - \lambda = z$ for $\lambda = 0.01$. Using a standard CF expansion:

$$\mathrm{Tan}(z^2) - \lambda = -\lambda + \frac{z^2}{1-} \frac{z^4}{3-} \frac{z^4}{5-} \frac{z^4}{7-} \cdots = z$$

Starting with $z = 0.5$, eight iterative steps give $\alpha = -0.00990195\cdots$, to fourteen decimal places.

Example 3 : $F(z) = \prod_{j=1}^{\infty} (1 - \frac{\sin z}{2 \cdot 4^j})$. Beginning with $z = 1$, we obtain $\alpha = 0.82701183\cdots$ to ten digits after sixteen steps.

Example 4 : $F(z) = \frac{1}{2^3 + z} + \frac{1}{2^4 + z} + \frac{1}{2^5 + z} + \cdots$. Starting with $z = 0.5$, we obtain $\alpha = 0.2450255\cdots$ to ten digits.

Reference:

[1] J. Gill, The Use of the Sequence $F_n(z) = f_n \circ \cdots \circ f_1(z)$ in Computing Fixed Points of Continued Fractions, Products, and Series, *Appl. Numer. Math.* **8** (1991) 469-476